19 Circles

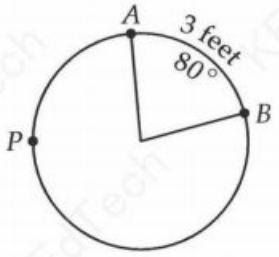
First, the basics. Given a circle with a radius of length r,

- the diameter of the circle is $2r$
- the area of the circle is πr^2
- the circumference of the circle is $2\pi r$

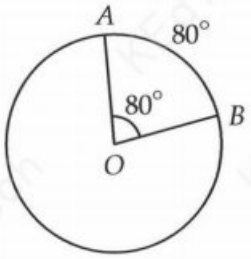
Arc Length vs. Arc Measure

An arc is a portion of a circle. Every arc has an arc length and an arc measure. Arc length is the actual distance along the circle from one point to another (e.g. 3 feet). Arc measure is the number of degrees of rotation along the circle from one point to another (e.g. 80°). A full rotation is 360°.

Minor arcs, such as arc AB below, have a measure of less than 180° and are named using two letters. Major arcs, such as arc APB below, are named using three letters.



The measure of an arc is equal to the measure of the central angle that intercepts it. In the circle below, arc AB has the same measure as central angle AOB.



By definition, a central angle of a circle is an angle whose vertex is at the center of the circle and whose legs are radii. For any given arc, the central angle that intercepts it is sometimes referred to as the arc's **associated angle**. In the example above, angle AOB is arc AB's associated angle.

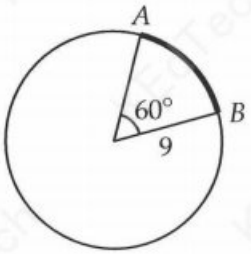
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The length of an arc can be found by using the formula

$$\frac{\theta}{360} \times 2\pi r$$

where θ is the degree measure of the central angle that intercepts the arc and r is the radius. The expression $\frac{\theta}{360}$ represents the fraction of the circle's circumference constituted by the arc, and $2\pi r$ is the circle's circumference.

In the circle below, arc AB constitutes $\frac{60^\circ}{360^\circ} = \frac{1}{6}$ of the circumference. Therefore, its length is $\frac{1}{6} \times 2\pi(9) = 3\pi$ units.



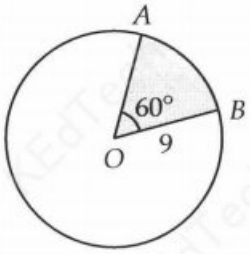
Area of a Sector

A sector is a "slice" of a circle. The area of a sector is given by

$$\frac{\theta}{360} \times \pi r^2$$

where θ is the degree measure of the central angle that encloses the sector and r is the radius. The expression $\frac{\theta}{360}$ represents the fraction of the circle's area constituted by the sector, and πr^2 is the circle's area.

In the circle below, sector OAB constitutes $\frac{60^\circ}{360^\circ} = \frac{1}{6}$ of the area. Therefore, its area is $\frac{1}{6} \times \pi(9)^2 = \frac{27}{2}\pi$ square units.

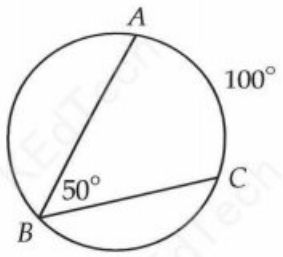


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Angles Inscribed in a Circle

An inscribed angle of a circle is an angle whose vertex lies on the circle and whose legs are chords (line segments whose endpoints both lie on the circle).

The measure of an inscribed angle is half the measure of the arc it intercepts. In the circle below, chords $\overline{AB}$ and $\overline{BC}$ make up inscribed angle ABC, which has half the measure of arc AC.

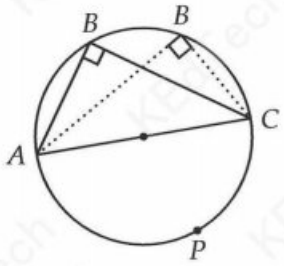


Angles Inscribed in a Semicircle

A semicircle is a half of a circle. We obtain an angle inscribed in a semicircle by designating a point on the circle as the vertex and drawing line segments to the endpoints of a diameter.

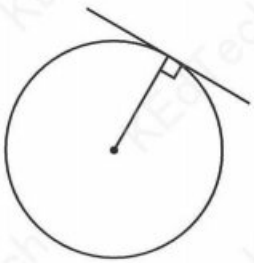
Because angles inscribed in a semicircle always intercept half a circle (180°), they always measure 90° (half of 180°).

In the circle below, $\overline{AC}$ is a diameter and angle ABC is inscribed in semicircle ABC. Because angle ABC intercepts arc APC (180°), angle ABC must be a right angle. Point B could be moved to any distinct location on the circle and angle ABC would still be 90°.



Radius Drawn to a Tangent Line

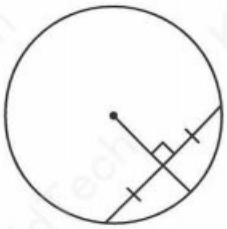
A line is tangent to a circle if it touches the circle at exactly one point. A radius drawn to that point of tangency is perpendicular to the line:



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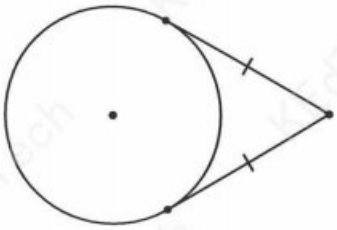
Radius that Bisects a Chord

A radius that splits a chord into two equal parts (bisects a chord) is perpendicular to that chord. The converse is also true: a radius that is perpendicular to a chord bisects that chord.

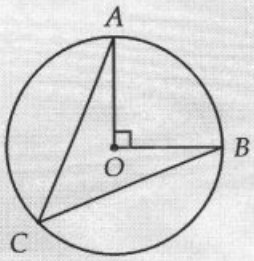


Tangent Segments to a Circle

Tangent segments to a circle drawn from the same external point are congruent.



EXAMPLE 1:



The circle shown has center O, and points A, B, and C lie on the circle. Angle AOB is a right angle. What is the measure, in degrees, of angle ACB ?

A) 15 B) 30 C) 45 D) 60

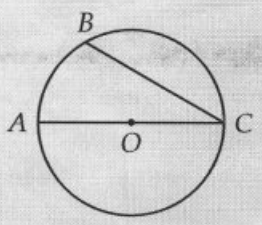
Since central angle AOB ($90°$) intercepts arc AB, arc AB also measures $90°$. Since angle ACB is an inscribed angle that intercepts arc AB, its measure must be half the measure of arc AB: $90° \div 2 = 45°$. Answer $\boxed{(C)}$.

EXAMPLE 2: A circle has radius of length 9 inches. The length of arc AB on this circle is 4π inches. What is the measure, in degrees, of arc AB ?

The circumference of the circle is $2\pi r = 2\pi(9) = 18\pi$ inches, which means arc AB constitutes $\frac{4\pi}{18\pi} = \frac{2}{9}$ of the circle. Given that a full circle measures $360°$, arc AB must have a measure of $\frac{2}{9} \times 360° = \boxed{80}$ degrees.

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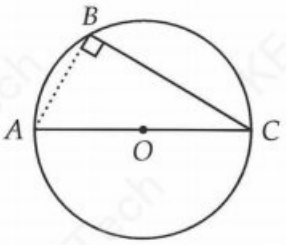
EXAMPLE 3:



In the circle shown, point O is the center, and diameter $\overline{AC}$ has a length of 10 inches. The measure of arc AB is $60°$. What is the length, in inches, of $\overline{BC}$?

A) 6 B) $4\sqrt{3}$ C) 8 D) $5\sqrt{3}$

First, the line that's begging to be drawn is $\overline{AB}$. Since $\overline{AC}$ is a diameter, arc ABC is a semicircle in which angle ABC is inscribed. Angle ABC must therefore be a right angle.



Since arc AB measures $60°$, inscribed angle ACB must measure $60° \div 2 = 30°$. Thus, triangle ABC is a 30–60–90 triangle with hypotenuse $\overline{AC}$ of length 10 inches.

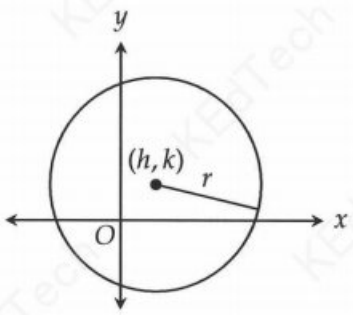
In a 30–60–90 triangle, the shorter leg (e.g. $\overline{AB}$) is half as long as the hypotenuse and the longer leg (e.g. $\overline{BC}$) is $\sqrt{3}$ times longer than the shorter leg. Therefore, the length of side $\overline{AB}$ is $10 \div 2 = 5$ inches, and the length of side $\overline{BC}$ is $5\sqrt{3}$ inches. Answer $\boxed{(D)}$.

Equation of a Circle in the xy-plane

The standard form equation of a circle in the xy-plane is

$$(x-h)^2 + (y-k)^2 = r^2$$

where (h,k) is the center of the circle and r is the length of its radius.



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Let's illustrate with a quick example. Let's say we have the following equation of a circle:

$$(x+6)^2+(y-5)^2=64$$

This equation is in standard form with $h=-6$, $k=5$, and $r=\sqrt{64}=8$. Therefore, the circle is centered at $(-6,5)$ and has a radius of 8.

When finding the center of a circle, the foolproof method I like to use is to ask myself what values make the expressions inside the parentheses equal to zero. *What value of x makes $x+6$ equal to zero? And what value of y makes $y-5$ equal to zero?* Well, $x=-6$ and $y=5$. Therefore, the center is at $(-6,5)$.

EXAMPLE 4: The equation $x^2+y^2=9$ represents a circle in the xy-plane. A new circle is drawn whose center is 8 units down from the center of this circle and whose radius is 7 units longer than the radius of this circle. Which of the following is an equation of this new circle?

A) $x^2+(y-8)^2=16$
B) $x^2+(y+8)^2=16$
C) $x^2+(y-8)^2=100$
D) $x^2+(y+8)^2=100$

The given circle has center $(0,0)$ and a radius of $\sqrt{9}=3$. The new center is 8 units down, so its coordinates must be $(0,0-8)=(0,-8)$. The new radius is 7 units longer, so its length must be $3+7=10$. The equation of the new circle is then

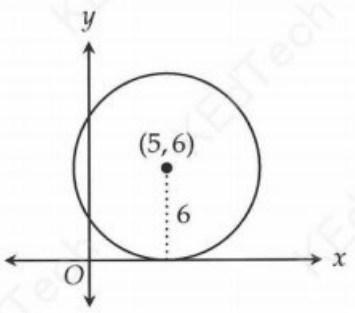
$$(x-0)^2+(y-(-8))^2=10^2$$
$$x^2+(y+8)^2=100$$

Answer $\boxed{(D)}$.

EXAMPLE 5: A circle in the xy-plane has center $(5,6)$ and intersects the x-axis at exactly one point. Which of the following equations represents this circle?

A) $(x-5)^2+(y-6)^2=25$
B) $(x+5)^2+(y+6)^2=25$
C) $(x-5)^2+(y-6)^2=36$
D) $(x+5)^2+(y+6)^2=36$

The circle's center is 5 units to the right of the y-axis and 6 units above the x-axis. For the circle to intersect the x-axis at exactly one point (i.e. be tangent to the x-axis), it must have a radius that is equal to the center's distance from the x-axis. Therefore, the circle must have a radius of 6.



The equation of the circle is then $(x-5)^2+(y-6)^2=36$. Answer $\boxed{(C)}$.

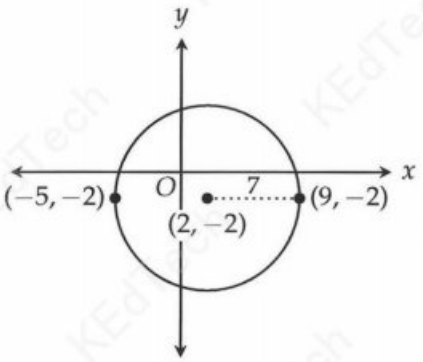
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EXAMPLE 6: A circle in the xy-plane has a diameter with endpoints $(-5, -2)$ and $(9, -2)$. Which of the following equations represents this circle?

A) $(x-2)^2+(y+2)^2=49$ B) $(x-2)^2+(y+2)^2=196$
C) $(x-4)^2+(y+2)^2=49$ D) $(x-4)^2+(y+2)^2=196$

The center of the circle is at the midpoint of the diameter's endpoints. The midpoint between two points can be found by averaging the x-coordinates and averaging the y-coordinates. Thus, the center is at $\left(\frac{-5+9}{2}, \frac{-2+(-2)}{2}\right) = (2, -2)$.

Since the endpoints of the given diameter have the same y-coordinates, the length of the diameter is just the distance between the x-coordinates: $9-(-5)=14$. The radius is then $14 \div 2 = 7$, and the equation of the circle is $(x-2)^2+(y+2)^2=49$.



Answer $\boxed{(A)}$.

Completing the Square

So far, the equations of circles we've dealt with have all been in standard form. But we can also get equations of circles that look like $x^2-10x+y^2-y=-5$. To convert these raw equations into standard form, we need to know how to complete the square. **Completing the square** is a technique used to convert quadratic expressions such as x^2+6x+2 into the form $(x+a)^2+b$ (i.e. vertex form, remember?).

How to Complete the Square

Let's say we need to complete the square for the following quadratic expression:

$$x^2+6x+2$$

Step 1) We know what the end result should look like. Create a template for it with a blank space:

$$(x\quad)^2+2$$

Step 2) Divide the coefficient of x by 2. Put the resulting number inside the parentheses and remember it for the next step. In this case, the coefficient of x is 6, and 6 divided by 2 is 3, so we put positive 3 inside the parentheses and remember it for the next step:

$$(x+3)^2+2$$

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Step 3) Square the number you remembered from the previous step. Subtract the result outside the parentheses. In this case, $3^2 = 9$, so we subtract 9 outside the parentheses:

$$(x+3)^2 + 2 - 9$$

Step 4) Combine like terms:

$$(x+3)^2 - 7$$

Step 5) And we're done! Let's check our work. Expanding $(x+3)^2 - 7$ gives $x^2 + 6x + 9 - 7 = x^2 + 6x + 2$, which is the expression we started with.

Now let's convert the equation of the circle given earlier into standard form by completing the square twice, once for the x's and once for the y's. The equation was

$$x^2 - 10x + y^2 - y = -5$$

Let's start with x and ignore the y's.

$$\underline{(x\quad)^2} + y^2 - y = -5$$
$$\underline{(x-5)^2 - 25} + y^2 - y = -5$$

Since the coefficient of x is -10 and half of -10 is -5, we put -5 inside the parentheses and subtract $(-5)^2 = 25$ outside the parentheses. Now we complete the square a second time for y:

$$(x-5)^2 - 25 + \underline{y^2 - y} = -5$$
$$(x-5)^2 - 25 + \underline{(y\quad)^2} = -5$$
$$(x-5)^2 - 25 + \underline{\left(y - \frac{1}{2}\right)^2 - \frac{1}{4}} = -5$$

Since the coefficient of y is -1 and half of -1 is $-\frac{1}{2}$, we put $-\frac{1}{2}$ inside the parentheses and subtract $\left(-\frac{1}{2}\right)^2 = \frac{1}{4}$ outside the parentheses. Combining like terms, we get

$$(x-5)^2 - 25 + \left(y - \frac{1}{2}\right)^2 - \frac{1}{4} = -5$$
$$(x-5)^2 + \left(y - \frac{1}{2}\right)^2 = 25 + \frac{1}{4} - 5$$
$$(x-5)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{81}{4}$$

And voilà, the equation is now in standard form. The circle is centered at $\left(5, \frac{1}{2}\right)$ and has a radius of $\sqrt{\frac{81}{4}} = \frac{9}{2}$.

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Completing the square may seem tedious, but it's actually not. You don't need to rewrite the entire equation for each step (we only do so for illustrative purposes). Normally, you would

$$\underbrace{(x-5)^2-25}_{\text{write this part first}}+\underbrace{\left(y-\frac{1}{2}\right)^2-\frac{1}{4}}_{\text{and then write this part second}}=-5$$

all on one line. Then combine like terms.

EXAMPLE 7:

$$9x^2+9y^2+18x-36y=4$$

The graph of the given equation in the xy-plane is a circle. What is the length of the circle's radius?

Before we can convert the given equation into standard form, we need to make the equation easier to work with. Let's first divide both sides by 9 so that the coefficients of x^2 and y^2 are 1:

$$x^2+y^2+2x-4y=\frac{4}{9}$$

Now let's rearrange the terms so that the x's are together and the y's are together:

$$x^2+2x+y^2-4y=\frac{4}{9}$$

Now we're ready to complete the square, first for the x's and then for the y's:

$$\underline{(x\quad)^2}+y^2-4y=\frac{4}{9}$$

$$\underline{(x+1)^2-1}+y^2-4y=\frac{4}{9}$$

$$(x+1)^2-1+\underline{(y\quad)^2}=\frac{4}{9}$$

$$(x+1)^2-1+\underline{(y-2)^2-4}=\frac{4}{9}$$

$$(x+1)^2+(y-2)^2=1+4+\frac{4}{9}$$

$$(x+1)^2+(y-2)^2=\frac{49}{9}$$

Now that the equation is in standard form, we can see that the radius of the circle is $\sqrt{\frac{49}{9}}=\boxed{\frac{7}{3}}$.

Desmos Note: If completing the square is a struggle for you, don't worry too much about it. It's not tested very often on the SAT, and when it is, you can usually just solve the problem by using Desmos, which can graph the equations of circles. For example, let's say you're given the equation of a circle and asked to find its radius. Instead of completing the square, you could graph the equation and calculate the radius from the coordinates of a few select points on the graph.

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EXERCISE 1: Answers for this exercise start on page 388.

1

What is the circumference of the circle in the xy-plane with equation $(x+7)^2+(y-5)^2=100$?

A) 10π
B) 20π
C) 100π
D) 200π

2

Point C lies on a circle with center O and diameter $\overline{AB}$. The measure of angle BAC is $47°$. What is the measure, in degrees, of arc BC ?

3

Which of the following equations represents a circle in the xy-plane with a radius of length 4 and center at $(0,1)$?

A) $x^2+(y-1)^2=4$
B) $(x-1)^2+y^2=4$
C) $x^2+(y-1)^2=16$
D) $(x-1)^2+y^2=16$

4

A circle in the xy-plane has the equation $(x-2)^2+(y-9)^2=169$. Which point (x,y) lies on this circle?

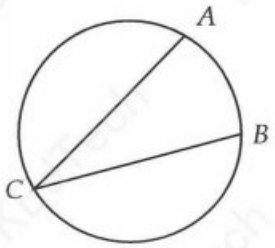
A) $(7,-3)$
B) $(7,-1)$
C) $(7,1)$
D) $(7,3)$

5

Circle A has an area of 32π and circle B has an area of 384π. The radius of circle B is how many times longer than the radius of circle A?

A) $2\sqrt{3}$
B) $3\sqrt{2}$
C) 6
D) 12

6



Note: Figure not drawn to scale.

Points A, B, and C lie on the circle shown. The measure of angle ACB is $30°$. The length of arc AB is what fraction of the circumference of the circle?

A) $\frac{1}{3}$
B) $\frac{1}{4}$
C) $\frac{1}{6}$
D) $\frac{1}{12}$

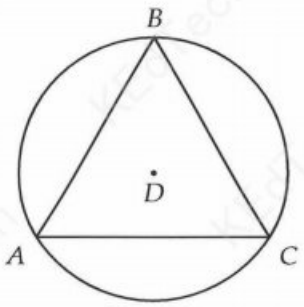
7

A circle in the xy-plane has the equation $x^2+y^2=9$. The line $y=0$ intersects this circle at two points. Which of the following is one of the points of intersection?

A) $(-3,0)$
B) $(0,-3)$
C) $(0,0)$
D) $(0,3)$

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8



The circle shown has center D and area 36π. Equilateral triangle ABC is inscribed in this circle. What is the length of arc AB ?

A) 2π
B) 3π
C) 4π
D) 6π

9

Point O is the center of a circle with a radius of 10 feet. The measure of arc CD on this circle is $72°$. What is the length, in feet, of arc CD ?

A) 2π
B) 4π
C) 5π
D) 20π

10

Circle A in the xy-plane has the equation $(x-1)^2+(y+2)^2=64$. Circle B is obtained by shifting circle A up 7 units. Which of the following equations represents circle B?

A) $(x-1)^2+(y-9)^2=64$
B) $(x-1)^2+(y+9)^2=64$
C) $(x-1)^2+(y-5)^2=64$
D) $(x-1)^2+(y+5)^2=64$

11

Point P is the center of a circle. The measure of arc QR on this circle is $144°$. What is the measure, in degrees, of angle PQR ?

12

A circle in the xy-plane has the equation $(x+4)^2+(y+6)^2=k$, where k is a constant. This circle intersects the x-axis at exactly one point. What is the value of k ?

13

A circle in the xy-plane has a diameter with endpoints $(-8,-5)$ and $(-8,5)$. Which equation represents this circle?

A) $(x+8)^2+(y-5)^2=100$
B) $(x+8)^2+y^2=100$
C) $(x+8)^2+(y-5)^2=25$
D) $(x+8)^2+y^2=25$

14

$$x^2+2x+y^2-8y=64$$

The graph of the given equation in the xy-plane is a circle. What is the length of the circle's radius?

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EXERCISE 2: Answers for this exercise start on page 389.

1

The point $(-2,0)$ is the center of a circle in the xy-plane. The circle has an area of 49π. Which of the following equations represents this circle?

A) $(x-2)^2+y^2=7$

B) $(x+2)^2+y^2=7$

C) $(x-2)^2+y^2=49$

D) $(x+2)^2+y^2=49$

2

The equation $(x+3)^2+(y-6)^2=9$ represents circle A. Circle B is obtained by shifting circle A to the right by 5 units in the xy-plane. What are the coordinates of the center of circle B?

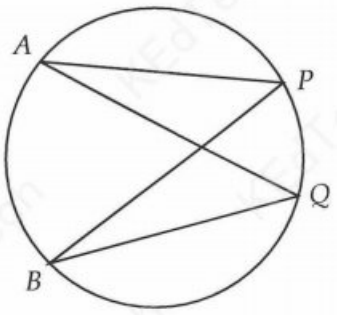
A) $(2,6)$

B) $(2,-6)$

C) $(8,6)$

D) $(8,-6)$

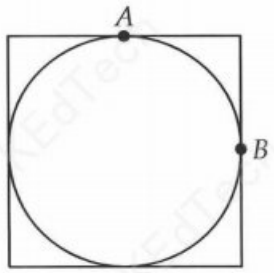
3



Note: Figure not drawn to scale.

Angles APB and AQB are inscribed in the circle shown. The measure of angle APB is $40°$. What is the measure, in degrees, of angle AQB?

4



In the figure, a circle is inscribed in a square with a side length of 14 inches. What is the length, in inches, of arc AB?

A) $\frac{7}{4}\pi$

B) $\frac{7}{2}\pi$

C) 7π

D) $\frac{49}{4}\pi$

5

The center of a circle in the xy-plane is at $(1,-1)$ and the point $(-5,7)$ lies on the circle. Which of the following equations represents this circle?

A) $(x+1)^2+(y-1)^2=10$

B) $(x-1)^2+(y+1)^2=10$

C) $(x+1)^2+(y-1)^2=100$

D) $(x-1)^2+(y+1)^2=100$

6

A circle has center O, and points M and N lie on the circle. The measure of arc MN is $24°$ and the length of arc MN is 8π centimeters. What is the length, in centimeters, of the radius of the circle?

A) 15

B) 45

C) 60

D) 120

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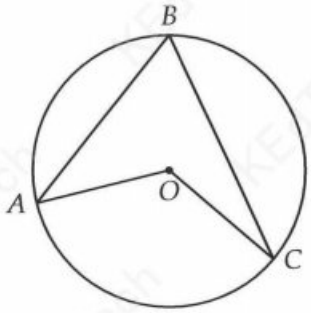
7

Triangle ABC is inscribed in a circle. The measure of angle A is equal to the measure of angle C. The measure of arc AC is $48°$. What is the measure, in degrees, of arc AB ?

8

In the xy-plane, circle A is defined by the equation $(x-3)^2+(y-4)^2=16$, and circle B is defined by the equation $(x-3)^2+(y-4)^2=n$, where n is a constant. The diameter of circle B is 48 units longer than the diameter of circle A. What is the value of n ?

9



The circle shown has center O, and points A, B, and C lie on the circle. The measure of angle BAO is $38°$ and the measure of angle BCO is $25°$. What is the measure, in degrees, of arc AC ?

10

Which of the following equations represents a circle in the xy-plane that intersects the line $x=5$ at exactly one point?

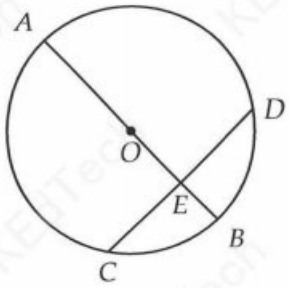
A) $(x-1)^2+(y-7)^2=4$

B) $(x-5)^2+(y-1)^2=4$

C) $(x-7)^2+(y-5)^2=4$

D) $(x-13)^2+(y-13)^2=4$

11



Note: Figure not drawn to scale.

The circle shown has center O and a radius of length 16. Diameter $\overline{AB}$ bisects $\overline{CD}$ at point E, and $CD=28$. If the length of $\overline{OE}$ can be written in the form $2\sqrt{k}$, where k is an integer, what is the value of k ?

12

In the xy-plane, a circle with a radius of 1 has its center at $(9,-12)$. This circle can be represented by the equation $x^2+y^2+ax+by+c=0$, where a, b, and c are constants. What is the value of $a+b+c$?

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13

A circle in the xy-plane is defined by the equation $(x-h)^2+(y-k)^2=144$, where h and k are constants. The point $(8,27)$ lies on this circle. Which of the following is a possible value for k ?

A) -8
B) 12
C) 39
D) 43

14

In the xy-plane, the points $(-4,3)$ and $(8,3)$ are endpoints of a diameter of a circle. An equation of this circle is $(x-a)^2+(y-b)^2=c^2$, where a, b, and c are positive constants. What is the value of $a+b+c$?

15

A circle has center A, and points B and C lie on the circle. Line segment BD is tangent to the circle at point B, line segment CD is tangent to the circle at point C, and the length of line segment BD is 63 units. If the circle has a radius of 16 units, what is the area, in square units, of quadrilateral $ABCD$?

16

$$x^2+y^2+7x+y=-\frac{7}{2}$$

The given equation represents a circle in the xy-plane. What is the length of the radius of the circle?

17

In the xy-plane, a circle has center D with coordinates $(m,16)$, where m is a constant. Points E and F lie on the circle. Point E lies on the circle and has coordinates $(m+4\sqrt{10},34)$, and point F has coordinates $(n,16)$, where n is a constant greater than m. What is the area, in square units, of triangle DEF ?

A) 198
B) $72\sqrt{10}$
C) 242
D) $99\sqrt{10}$

18

A circle in the xy-plane has the equation $(x-2)^2+(y-2)^2=25$. Line m is tangent to this circle at the point $(5,6)$. What is the y-coordinate of the y-intercept of line m ?

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