22

Area, Perimeter, & Volume

Since we've already discussed polygons, triangles, and circles in previous chapters, the focus of this chapter will be on geometrical shapes and concepts we haven't covered yet.

Squares, rectangles, and parallelograms

Square



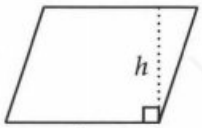
Area $= s^2$
Perimeter $= 4s$

Rectangle



Area $= lw$
Perimeter $= 2l + 2w$

Parallelogram



Area $= bh$

Parallelograms are quadrilaterals in which opposite sides are parallel and equal in length. Rectangles are parallelograms that have four right angles. Squares are rectangles with four sides of equal length.

EXAMPLE 1: A square has a perimeter of 76 inches. What is the area of this square, in square inches?

A) 19 B) 361 C) 1,444 D) 5,776

The square must have a side length of $76 \div 4 = 19$ inches. Therefore, its area is $s^2 = 19^2 = 361$ square inches. Answer $\boxed{(B)}$.

K Education & Technology

EXAMPLE 2: A rectangle has a width that is 3 inches less than its length. If the area of the rectangle is 108 square inches, what is the perimeter, in inches, of the rectangle?

Let the length of the rectangle be l. Its width w is then $l-3$. Since the area of the rectangle is 108 in^2,

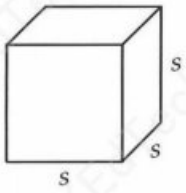
$$\begin{aligned} lw &= 108 \\ l(l-3) &= 108 \\ l^2-3l-108 &= 0 \\ (l-12)(l+9) &= 0 \end{aligned}$$

Since the length of a rectangle has to be positive, $l=12$ and $w=l-3=12-3=9$. Finally, the perimeter is $2l+2w=2(12)+2(9)=\boxed{42}$ inches.

Right prisms and cylinders

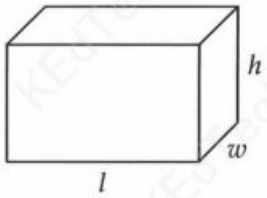
In geometry, a **solid** is a 3-dimensional shape. A **prism** is a solid with flat sides and two identical ends (often called bases). A **right prism** is a prism in which the sides are perpendicular to the bases. In other words, a right prism "stands up straight." The prisms you encounter on the SAT will be right prisms:

Cube



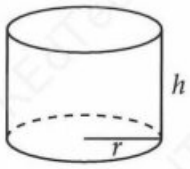
Volume $= s^3$
Surface area $= 6s^2$

Right Rectangular Prism



Volume $= lwh$
Surface area $= 2lw+2lh+2wh$

Right Circular Cylinder



Volume $= \pi r^2 h$

The volume of any right prism can be found using the following formula:

$$\text{Volume} = \text{Area of base} \times \text{Height}$$

This formula explains why the volume of a cube is s^3 (base area of s^2 and height of s), the volume of a right rectangular prism is lwh (base area of lw and height of h), and the volume of a right circular cylinder is $\pi r^2 h$ (base area of πr^2 and height of h). Note that a cylinder is technically not a prism by definition but the concept still applies.

The **surface area** of a solid is the total area that the surface of the solid occupies. Since a cube has 6 faces and each face has an area of s^2, the surface area of a cube is $6s^2$. A right rectangular prism, on the other hand, has 2 faces each with an area of lw, 2 faces each with an area of lh, and 2 faces each with an area of wh, which together yield a surface area of $2lw+2lh+2wh$. The surface area of a right circular cylinder is derived in Example 4.

K Education & Technology

EXAMPLE 3: A right rectangular prism has a length of 15 inches and a width of 18 inches. If the volume of the prism is 7,020 cubic inches, what is the surface area, in square inches, of this prism?

First, we need to solve for the height h. Since the volume of the prism is 7,020 cubic inches,

$$\begin{aligned} lwh &= 7{,}020 \\ (15)(18)h &= 7{,}020 \\ h &= 7{,}020 \div (15 \cdot 18) = 26 \text{ inches} \end{aligned}$$

The surface area of the prism is then $2lw + 2lh + 2wh = 2(15)(18) + 2(15)(26) + 2(18)(26) = \boxed{2256}$ in^2.

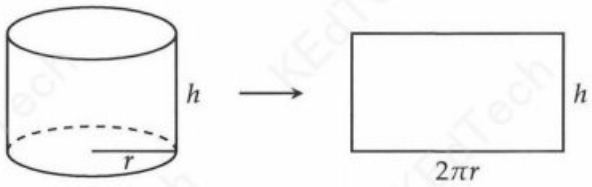
EXAMPLE 4: A right circular cylinder has a radius of 4 centimeters (cm) and a height of 5 cm. What is the surface area, in cm^2, of this cylinder?

A) 40π B) 60π C) 72π D) 81π

For the sake of learning, let's first derive the formula for the surface area of a cylinder with radius r and height h. Then we'll apply the formula to this question.

The surface area of a cylinder consists of the areas of its two bases and the area of the curved surface that goes around the cylinder. Since the two bases are circles and each circle has an area of πr^2, the two bases contribute a total surface area of $2\pi r^2$. That's the easy part.

To find the area of the curved surface that goes around the cylinder, we have to imagine that the curved surface is unraveled like wrapping paper:



Unraveling the curved surface results in a rectangle with a length equal to the cylinder's base circumference, $2\pi r$, and a width equal to the height of the cylinder, h. The area of this rectangle is the area of the curved surface: $lw = 2\pi rh$.

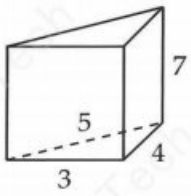
Summing up the areas of the bases and the area of the curved surface, we get the following formula:

$$\text{Surface area of a cylinder} = 2\pi r^2 + 2\pi rh$$

Applying this formula to the cylinder given in the question, we get a surface area of $2\pi(4)^2 + 2\pi(4)(5) = 32\pi + 40\pi = 72\pi$. Answer $\boxed{(C)}$.

K Education & Technology

EXAMPLE 5: A triangular prism has a height of 7 inches. The base of the prism is a right triangle with sides of length 3, 4, and 5 inches. What is the volume, in cubic inches, of the prism?

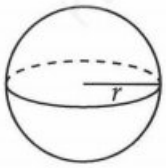


Since the base of the prism is a 3–4–5 right triangle (the 5-inch side being the hypotenuse), the area of the base is $\frac{1}{2}(\text{leg}_1)(\text{leg}_2) = \frac{1}{2}(3)(4) = 6$ square inches. The volume of the prism is then the area of the base multiplied by the height of the prism: $6 \times 7 = \boxed{42}$ cubic inches.

Spheres, Cones, and Pyramids

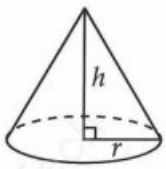
The following solids also appear on the SAT, but not as often as cubes, rectangular prisms, and cylinders. Even though the SAT gives you these formulas at the beginning of each math section, all the volume formulas presented in this chapter are worth memorizing.

Sphere



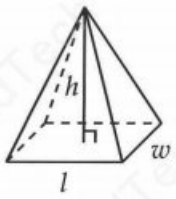
Volume $= \frac{4}{3}\pi r^3$

Cone



Volume $= \frac{1}{3}\pi r^2 h$

Square Pyramid



Volume $= \frac{1}{3}lwh$

Changes that affect area and volume

How does doubling both the length and the width of a rectangle affect its area? How does increasing the radius of a sphere by 50% affect its volume? The following examples will show you how to tackle these types of questions.

EXAMPLE 6: Circle A has a radius that is 30% longer than the radius of circle B. The area of circle A is k percent greater than the area of circle B. What is the value of k?

A) 9 B) 30 C) 36 D) 69

If we let r_a be the radius of circle A and r_b be the radius of circle B, then $r_a = 1.30r_b$ (if you don't understand why, see the Percent chapter). The area of circle A can then be expressed as

$$\pi r_a^2 = \pi(1.30r_b)^2 = \pi \cdot (1.30)^2 \cdot r_b^2 = (1.30)^2 \pi r_b^2 = 1.69\pi r_b^2$$

Note the use of parentheses when we substitute $1.30r_b$ for r_a in the first step. These parentheses are very important. In the following steps, we just rearrange the terms until there is a single coefficient, 1.69, in front of πr_b^2, the area of circle B.

This coefficient indicates that the area of circle A is 1.69 times the area of circle B. The area of circle A is therefore 69 percent greater than the area of circle B. Answer $\boxed{(D)}$. Here's another way to interpret this result—increasing the radius of a circle by 30% will increase its area by 69%.

K Education & Technology

EXAMPLE 7: Sphere P has a radius that is 5 times the radius of sphere Q. The volume of sphere P is how many times the volume of sphere Q?

If we let r_p be the radius of sphere P and r_q be the radius of sphere Q, then $r_p = 5r_q$. The volume of sphere P is then

$$\frac{4}{3}\pi r_p^3 = \frac{4}{3}\pi(5r_q)^3 = \frac{4}{3}\pi \cdot (5)^3 \cdot r_q^3 = (5)^3\left(\frac{4}{3}\pi r_q^3\right) = 125\left(\frac{4}{3}\pi r_q^3\right) = \boxed{125} \times \text{the volume of sphere Q}$$

Notice that 125 is just 5^3. The factor of 5 gets cubed because of the r^3 in the volume formula. Using this "shortcut," we can easily determine what happens in other scenarios. For example, increasing the radius of a sphere by a factor of 6 would result in a volume that is $6^3 = 216$ times larger. By reverse logic, a sphere whose volume is 216 times the volume of a smaller sphere must have a radius that is $\sqrt[3]{216} = 6$ times longer than that of the smaller sphere.

EXAMPLE 8: Cones A and B are right circular cones. Cone A has a radius that is $\frac{1}{2}$ the radius of cone B and a height that is 3 times the height of cone B. Cone B has a volume of 720 cubic inches. What is the volume of cone A, in cubic inches?

Let r_a and h_a be the radius and height of cone A, and let r_b and h_b be the radius and height of cone B. Based on the information given, $r_a = \frac{1}{2}r_b$ and $h_a = 3h_b$. The volume of cone A can then be expressed as

$$\frac{1}{3}\pi r_a^2 h_a = \frac{1}{3}\pi\left(\frac{1}{2}r_b\right)^2(3h_b) = \frac{1}{3}\pi \cdot \left(\frac{1}{2}\right)^2 \cdot r_b^2 \cdot 3 \cdot h_b = \left(\frac{1}{2}\right)^2 \cdot 3 \cdot \left(\frac{1}{3}\pi r_b^2 h_b\right) = \frac{3}{4}\left(\frac{1}{3}\pi r_b^2 h_b\right)$$

Therefore, the volume of cone A is $\frac{3}{4}$ the volume of cone B: $720 \times \frac{3}{4} = \boxed{540}$ in^3.

Similar polygons and solids

Two polygons are similar if their corresponding angles are congruent and their corresponding sides are in the same proportion. Two solids are similar if they are of the same type and their corresponding dimensions are in the same proportion. Similar polygons and similar solids are essentially different size versions of the same shape/solid.



Similar Trapezoids



Similar Rectangular Prisms

If two polygons are similar, then the ratio of their areas is equal to the **square** of the ratio of their corresponding side lengths. If two solids are similar, then the ratio of their volumes is equal to the **cube** of the ratio of their corresponding dimensions.

So let's say that triangles A and B are similar. If triangle A has side lengths that are 3 times the corresponding side lengths of triangle B (ratio of 3:1), then the area of triangle A is 9 times the area of triangle B (ratio of 3^2:1^2 = 9:1).

K Education & Technology

EXAMPLE 9: Polygon A is similar to polygon B. Each side of polygon A has a length of 15 units, and each side of polygon B has a length of 3 units. The area of polygon A is how many times the area of polygon B?

Polygon A has sides that are $15 \div 3 = 5$ times as long as the corresponding sides of polygon B. Therefore, the area of polygon A is $5^2 = \boxed{25}$ times the area of polygon B.

EXAMPLE 10: Rectangular prisms A and B are similar. Each edge length of prism B is 12 times the corresponding edge length of prism A. The volume of prism A is 42 cubic inches. What is the volume, in cubic inches, of prism B?

Since the edges of prism B are 12 times as long as the corresponding edges of prism A, the volume of prism B must be $12^3 = 1{,}728$ times the volume of prism A. The volume of prism B is therefore $1{,}728 \times 42 = \boxed{72576}$ cubic inches.

Density

Density is a substance's mass per unit of volume. Memorize the formula:

$$\text{Density} = \frac{\text{Mass}}{\text{Volume}}$$

Denser objects are heavier relative to their size.

EXAMPLE 11:

The volume of a sample of plastic is 1.5 cubic feet, and its mass is 126 pounds. What is the density, in pounds per cubic foot, of this sample?

A) 42 B) 84 C) 124.5 D) 189

$$\text{Density} = \frac{\text{Mass}}{\text{Volume}} = \frac{126 \text{ pounds}}{1.5 \text{ cubic feet}} = 84 \text{ pounds per cubic foot}$$

Answer $\boxed{(B)}$.

K Education & Technology

EXERCISE 1: Answers for this exercise start on page 398.

1

The mass of a foam block is 8 kilograms. The volume of the block is 20 cubic meters. What is the density, in kilograms per cubic meter, of the block?

A) 0.4
B) 2.5
C) 12
D) 28

2

A cube has a volume of 42,875 cubic inches. What is the surface area, in square inches, of the cube?

3

A right square pyramid has a height of 33 centimeters (cm) and a square base with a side length of 14 cm. What is the volume, in cm^3, of this pyramid?

4

A cylinder has a volume of 980π cubic feet and a height of 5 feet. What is the base diameter, in feet, of the cylinder?

5

Square A has an area of 64 square centimeters. The perimeter of square B is 4 times the perimeter of square A. What is the side length, in centimeters, of square B?

A) 16
B) 32
C) 64
D) 128

6

Quadrilateral X is similar to quadrilateral Y, where the longest side of quadrilateral X corresponds to the longest side of quadrilateral Y. The longest side of quadrilateral X has a length of 3 inches, and the longest side of quadrilateral Y has a length of 9 inches. The area of quadrilateral Y is how many times the area of quadrilateral X?

A) 3
B) 9
C) 27
D) 36

7

A right rectangular prism has a length of 16 inches, a width of 18 inches, and a height of 20 inches. What is the surface area, in square inches, of this prism?

K Education & Technology

8

The perimeter of square A is 144 inches greater than the perimeter of square B. The side length of square A is how many inches greater than the side length of square B?

A) 12
B) 36
C) 72
D) 144

9

A rectangle has a width of $(x - 8)$ units and a length of $(x + 5)$ units. If the rectangle has an area of 68 square units, what is the value of x ?

A) 9
B) 12
C) 15
D) 17

10

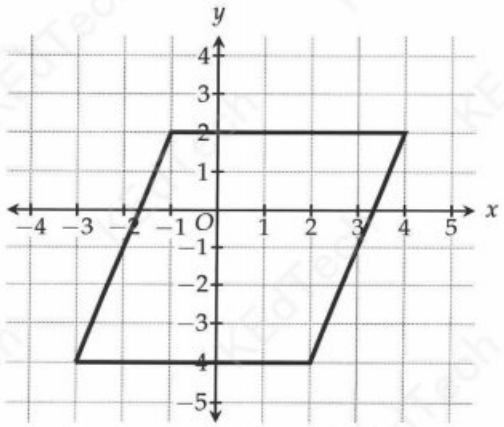
Parallelogram $ABCD$ is similar to parallelogram $PQRS$ such that A, B, C, and D correspond to P, Q, R, and S, respectively. The ratio of the length of side AB to the length of side PQ is 3 to 2. The area of $PQRS$ is 72 square units. What is the area, in square units, of $ABCD$?

A) 32
B) 108
C) 162
D) 324

11

The length of rectangle A is 54 times the length of rectangle B and the width of rectangle A is 45 times the width of rectangle B. The area of rectangle A is n times the area of rectangle B. What is the value of n ?

12



What is the area, in square units, of the parallelogram shown?

13

An object in the shape of a cube has a density of 390 grams per cubic centimeter. The object has a mass of 225 grams. To the nearest hundredth of a centimeter, what is the length of one edge of the object?

A) 0.58
B) 0.76
C) 0.83
D) 1.20

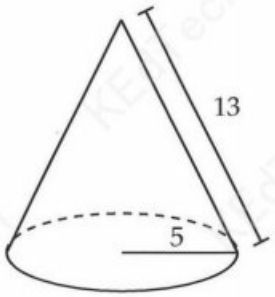
14

The volume of a sphere is equal to the volume of a cube with an edge length of 15 inches. To the nearest tenth of an inch, what is the length of the radius of the sphere?

A) 6.4
B) 9.3
C) 11.3
D) 13.6

K Education & Technology

15



The right circular cone shown has a radius of 5 inches and a slant height of 13 inches. The volume of this cone is $k\pi$ cubic inches. What is the value of k ?

16

Cylinders A and B are right circular cylinders. The radius of cylinder A is 14 times the radius of cylinder B, and the height of cylinder A is half the height of cylinder B. The volume of cylinder A is how many times the volume of cylinder B?

K Education & Technology

EXERCISE 2: Answers for this exercise start on page 399.

1

The area of square A is 576 times the area of square B. The perimeter of square A is how many times the perimeter of square B?

A) 6
B) 12
C) 24
D) 96

2

The surface area, in square inches, of a cube can be represented by the expression $24a^2$, where a is a positive constant. Which expression represents the volume, in cubic inches, of the cube?

A) $8a^3$
B) $16a^3$
C) $64a^6$
D) $216a^6$

3

A right circular cylinder is carved from a stone slab that has a density of 2.4 grams per cubic centimeter. The height of the cylinder is 4 centimeters, and the base of the cylinder has an area of 28 square centimeters. To the nearest whole number, what is the mass, in grams, of this cylinder?

A) 3
B) 17
C) 47
D) 269

4

Rectangular prism X is similar to rectangular prism Y. Each edge length of rectangular prism Y is 3 times the corresponding edge length of rectangular prism X. The volume of rectangular prism Y is 525 cubic meters. What is the volume, in cubic meters, of rectangular prism X?

5

$$f(x) = \pi x^3 + \pi x^2$$

The given function f represents the volume, in cubic feet, of a right circular cylinder with a radius of x feet. Which expression represents the height, in feet, of the cylinder?

A) $x + 1$
B) πx^3
C) x
D) $x^3 + 1$

6

Triangle ABC is similar to triangle XYZ such that A, B, and C correspond to X, Y, and Z, respectively. The area of triangle ABC is 16 times the area of triangle XYZ. If the length of side AB is 68 inches, what is the length, in inches, of side XY?

K Education & Technology

7

$$2xy + 100x + 100y$$

The given expression represents the surface area, in square centimeters, of a right rectangular prism with a length of x centimeters and a width of y centimeters. What is the height, in centimeters, of the rectangular prism?

A) 2
B) 25
C) 50
D) 100

8

$$A = x\left(\frac{75 - 2x}{2}\right)$$

The given equation represents the area A, in square inches, of a rectangle with a length of x inches. Which of the following is the best interpretation of 75 in this context?

A) The length of the rectangle is 75 inches greater than its width.
B) The sum of the length and the width of the rectangle is 75 inches.
C) The area of the rectangle cannot be greater than 75 square inches.
D) The perimeter of the rectangle is 75 inches.

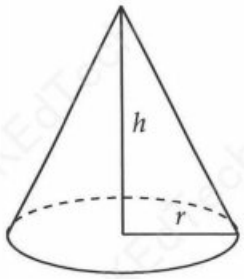
9

A rectangular image on a screen has an area of 464 square centimeters. The image is edited so that its length is 25% greater than its original length and its width is 10% less than its original width. What is the area, in square centimeters, of the edited image?

10

A rectangle has a width that is 6 inches less than 5 times its length. The area of the rectangle is 287 square inches. What is the length of the rectangle, in inches?

11



Cone M is a right circular cone with a radius of r and height of h, as shown. The volume of cone N (not shown) is 144 times the volume of cone M. Which of the following could represent the radius of cone N in terms of r and the height of cone N in terms of h ?

A) radius $= 4r$, height $= 6h$
B) radius $= 4r$, height $= 36h$
C) radius $= 6r$, height $= 4h$
D) radius $= 6r$, height $= 12h$

K Education & Technology

12

A rectangular prism has a width that is 6 inches greater than its length and a height that is 12 inches less than its width. The equation $y = x(x-6)(x+6)$ represents this situation, where y is the volume, in cubic inches, of the rectangular prism and $y > 0$. Which of the following is the best interpretation of $x - 6$ in this context?

A) The length of the rectangular prism, in inches
B) The width of the rectangular prism, in inches
C) The height of the rectangular prism, in inches
D) The difference between the width and the length of the rectangular prism, in inches

13

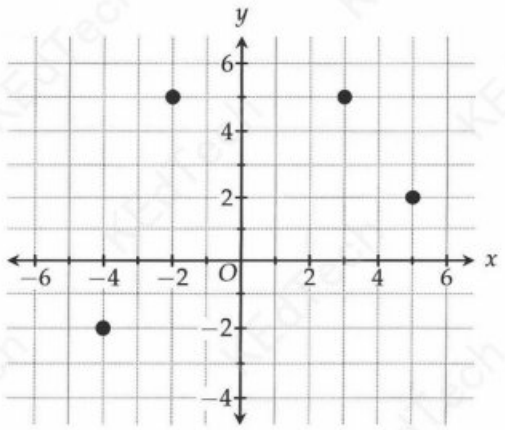
An artifact in the shape of a right circular cylinder has a volume of 112 cubic inches. A replica of the artifact is made in which the radius and height of the original artifact are each increased by 50%. What is the volume, in cubic inches, of the replica?

14

A sphere has a diameter of 10 centimeters. A solid right circular cylinder is inside the sphere, such that the two bases of the cylinder are distinct parallel cross-sections of the sphere. Each base has a diameter of 6 centimeters. To the nearest cubic centimeter, what is the volume of the space in the sphere not taken up by the cylinder?

A) 226
B) 241
C) 297
D) 354

15



A quadrilateral is formed by connecting the four points shown. What is the area, in square units, of this quadrilateral?

16

Identical rectangles X and Y each have an area of $\frac{49}{5}$ square meters and a perimeter of P meters. If one of the shorter sides of rectangle X is glued to one of the shorter sides of rectangle Y, the resulting rectangle has a perimeter of $\frac{14}{9}P$ meters. What is the value of P ?

K Education & Technology