20 Radians

A radian is simply another unit of angle measure. Just as we have feet and meters for distances and pounds and kilograms for weights, we have both degrees and radians for angles.

$$\pi \text{ radians} = 180^\circ$$

If you've never used radians before, don't be put off by the π. After all, it's just a number. We could've written

$$3.14 \text{ radians} \approx 180^\circ$$

instead, but 3.14 is only an approximation. Values are typically expressed in terms of π when radians are being used.

EXAMPLE 1: An angle has a measure of 45 degrees. What is the measure of this angle, in radians?

A) $\frac{\pi}{6}$ B) $\frac{\pi}{4}$ C) $\frac{\pi}{3}$ D) $\frac{\pi}{2}$

To convert from degrees to radians, we use the conversion factor $\frac{\pi \text{ radians}}{180^\circ}$.

$$45^\circ \times \frac{\pi \text{ radians}}{180^\circ} = \frac{\pi}{4} \text{ radians}$$

Answer $\boxed{(B)}$. Notice that the degree units (represented by the circle symbols) cancel out just as they should in any conversion problem.

EXAMPLE 2: An angle has a measure of $\frac{3\pi}{2}$ radians. What is the measure of the angle, in degrees?

This time, we're going from radians to degrees, so we should flip the conversion factor we used in Example 1:

$$\frac{3\pi}{2} \text{ radians} \times \frac{180^\circ}{\pi \text{ radians}} = \boxed{270}^\circ$$

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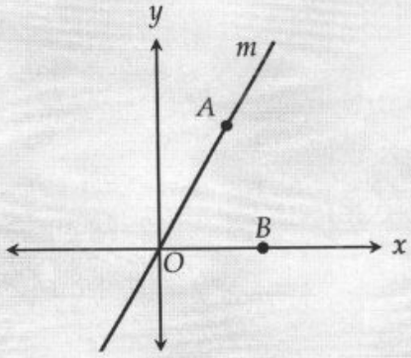
You might be wondering why we even need radians. Why not just stick with degrees? Is this another difference between the U.S. and the rest of the world, like it is with feet and meters? Nope. Some calculations are much simpler when the angles are in radians. For example, the length of an arc or the area of a sector can be found very easily when the central angle is in radians:

$$\frac{\theta}{2\pi} = \frac{AL}{2\pi r} \qquad \text{Arc Length} = r\theta \qquad \text{Area of a Sector} = \frac{1}{2}r^2\theta \qquad \frac{\theta}{2\pi} = \frac{\text{Area}}{\pi r^2}$$

$\theta \cdot r$ = Arc length $\qquad \frac{\theta \cdot r^2}{2}$ = Area

In these formulas, r is the length of the circle's radius and θ is the radian measure of the central angle. These formulas are nice to know, but you don't need to have them memorized for the SAT.

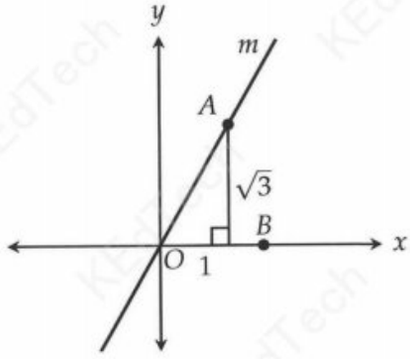
EXAMPLE 3:



In the xy-plane, point A lies on line m and point B lies on the x-axis as shown. If line m passes through the point $(0,0)$ and has a slope of $\sqrt{3}$, what is the measure, in radians, of angle AOB ?

A) $\frac{\pi}{6}$ B) $\frac{\pi}{5}$ C) $\frac{\pi}{4}$ D) $\frac{\pi}{3}$

We can draw a line segment from point A down to the x-axis to make a right triangle. Because the slope of line m is $\sqrt{3}$, the ratio of the height of this triangle to its base is always $\sqrt{3}$ to 1 (rise over run).



In other words, this right triangle has legs whose lengths are in a ratio of $\sqrt{3}$ to 1. It must therefore be a 30–60–90 triangle, in which the angle opposite the $\sqrt{3}$-side is 60°. The measure of angle AOB is then

$$60° \times \frac{\pi \text{ radians}}{180°} = \frac{\pi}{3} \text{ radians}$$

Answer (D).

$$\frac{\theta}{2\pi} = \frac{S}{\pi r^2} \qquad S = \frac{r^2 \cdot \theta}{2}$$

$180° = \pi$

$$\frac{\theta}{2\pi} = \frac{\text{length of arc}}{2\pi r}$$

$\theta \cdot r$ = length of arc.

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EXERCISE: Answers for this exercise start on page 392.

1

An angle has a measure of $\frac{17\pi}{12}$ radians. What is the measure of the angle, in degrees?

$180° = \pi$

$\frac{180°}{\pi} \times \frac{17\pi}{12} = \frac{180 \times 17}{12}$

$= 255°$

2

An angle has a measure of 225 degrees. What is the measure of this angle, in radians?

A) $\frac{3\pi}{4}$

B) $\frac{7\pi}{6}$

C) $\frac{5\pi}{4}$

D) $\frac{3\pi}{2}$

$225° \times \frac{\pi}{180°}$

3

The measure of angle A is $\frac{5\pi}{9}$ radians. The measure of angle B is $\frac{7\pi}{36}$ radians less than the measure of angle A. What is the measure of angle B, in degrees?

A) 35

B) 65

C) 100

D) 135

$A = 100°$

$\frac{7\pi}{36} =$

4

The measure of angle P is 162 degrees greater than the measure of angle R. How much greater is the measure of angle P than the measure of angle R, in radians?

A) $\frac{9}{20}\pi$

B) $\frac{9}{10}\pi$

C) $\frac{10}{9}\pi$

D) $\frac{20}{9}\pi$

5

An arc of a circle has a measure of 3.4 radians. What is the measure, in degrees, of this arc, to the nearest degree? (Disregard the degree sign when entering your answer. For example, if your answer is 36°, enter 36)

$\pi = 3.14$

$180° = \pi$

$3.4 \times 1 = 3.4 \times \frac{180°}{\pi}$

$1 \text{ rad} = \frac{180°}{\pi} \approx 194.81°$

$= 195$

6

A circle has center O, and points A and B lie on the circle. The length of arc AB is $\frac{1}{5}$ of the circumference of the circle. What is the measure, in radians, of angle AOB ?

A) $\frac{\pi}{10}$

B) $\frac{\pi}{5}$

C) $\frac{2\pi}{5}$

D) $\frac{5\pi}{2}$

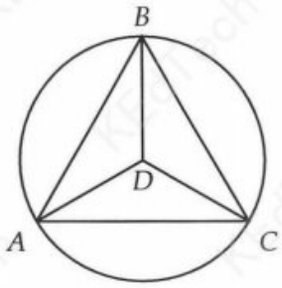
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7

An angle has a measure of 900 degrees. The number of radians in this angle can be written as $k\pi$, where k is a constant. What is the value of k ?

5

8



In the figure shown, equilateral triangle ABC is inscribed in a circle with center D. What is the measure, in radians, of angle ADB ?

A) $\frac{2\pi}{3}$

B) $\frac{3\pi}{4}$

C) $\frac{4\pi}{5}$

D) $\frac{5\pi}{6}$

9

Point O is the center of a circle. The measure of arc MN on this circle is $\frac{4\pi}{3}$ radians. The length of arc MN is what fraction of the circumference of the circle?

$\frac{\theta}{2\pi} = \frac{\text{The length of arc}}{2\pi r}$

θ 叫 ratio

$\frac{\frac{4\pi}{3}}{2\pi} = \frac{2\cancel{4\pi}}{3} \times \frac{1}{2\cancel{\pi}} = \frac{2}{3}$

10

A circle has center O, and points X and Y lie on the circle. The measure of angle OXY is $\frac{\pi}{9}$ radians. What is the measure, in radians, of angle XOY ?

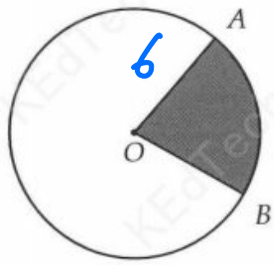
A) $\frac{5\pi}{18}$

B) $\frac{5\pi}{9}$

C) $\frac{2\pi}{3}$

D) $\frac{7\pi}{9}$

11



In the figure shown, a circle has center O and a radius of 6 inches. The area of the shaded sector is 10π square inches. What is the measure, in radians, of angle AOB ?

A) $\frac{2\pi}{5}$

B) $\frac{4\pi}{9}$

C) $\frac{5\pi}{9}$

D) $\frac{5\pi}{8}$

$\frac{\theta}{2\pi} = \frac{S}{\pi r^2}$

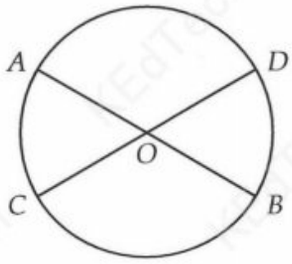
$S = \frac{\theta r^2}{2}$

$10\pi = \frac{\theta \cdot 6^2}{2}$

$\frac{\frac{5}{20\pi}}{6 \cdot 6} = \theta$ $\quad \theta = \frac{5\pi}{9}$

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12



Note: Figure not drawn to scale.

The circle shown has center O and diameters $\overline{AB}$ and $\overline{CD}$. The length of arc AD is three times the length of arc BD. What is the measure of angle AOD, in radians?

A) $\frac{\pi}{4}$

B) $\frac{\pi}{3}$

C) $\frac{2\pi}{3}$

D) $\frac{3\pi}{4}$

13

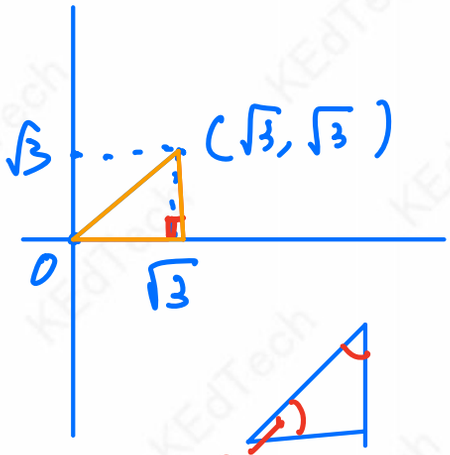
In the xy-plane, segment OA has endpoints $(0,0)$ and $(\sqrt{3},\sqrt{3})$. What is the measure, in radians, of the acute angle formed by segment OA and the line $y=0$?

A) $\frac{\pi}{6}$

B) $\frac{\pi}{4}$

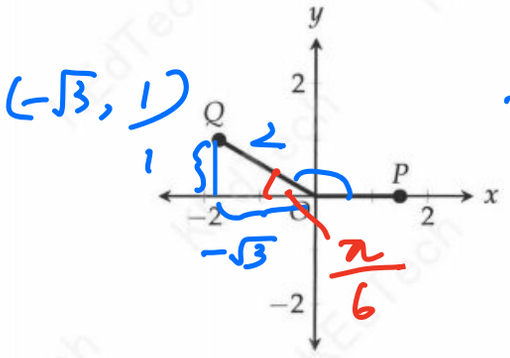
C) $\frac{\pi}{3}$

D) $\frac{\pi}{2}$



$\frac{\pi}{4} = 45°$

14



$\pi - \frac{\pi}{6} = \frac{5\pi}{6}$

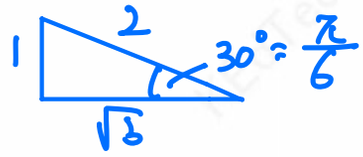
In the figure shown, the coordinates of point Q are $(-\sqrt{3},1)$. What is the measure, in radians, of angle POQ?

A) $\frac{2\pi}{3}$

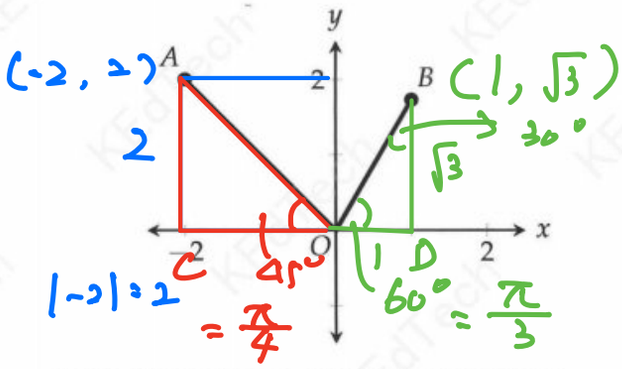
B) $\frac{3\pi}{4}$

C) $\frac{5\pi}{6}$

D) $\frac{11\pi}{12}$



15



In the figure shown, points A and B have coordinates $(-2,2)$ and $(1,\sqrt{3})$, respectively. What is the measure, in radians, of angle AOB?

A) $\frac{3}{8}\pi$

B) $\frac{5}{12}\pi$

C) $\frac{4}{9}\pi$

D) $\frac{7}{12}\pi$

$\angle AOB = \pi - \frac{\pi}{4} - \frac{\pi}{3}$

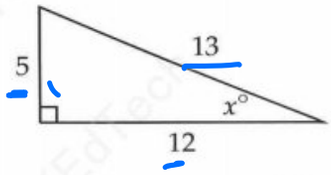
$= \pi - \frac{3\pi}{12} - \frac{4\pi}{12}$

$= \frac{12\pi - 7\pi}{12} = \frac{5\pi}{12}$

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21 Trigonometry

Let's use a 5–12–13 right triangle to illustrate the functions sine, cosine, and tangent—the three trigonometric functions you'll need to know:

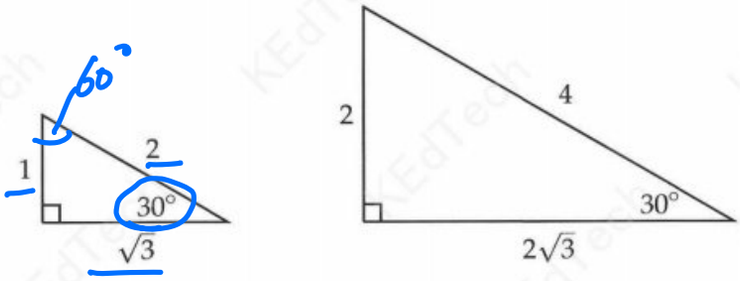


For the x° angle above,

$$\sin x^\circ = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{5}{13} \qquad \cos x^\circ = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{12}{13} \qquad \tan x^\circ = \frac{\text{opposite}}{\text{adjacent}} = \frac{5}{12}$$

where "opposite" refers to the length of the side opposite the x° angle and "adjacent" refers to the length of the side next to the x° angle (the one that's not the hypotenuse).

The outputs of trigonometric functions are just ordinary numbers, so don't let the strange names confuse you. They're just ratios. For instance, the sine of 30° ($\sin 30^\circ$) is always equal to $\frac{1}{2}$. Why? Let's take a look at the 30–60–90 triangles below.



As you can see, in the triangle on the left, $\sin 30^\circ = \frac{\text{opp}}{\text{hyp}} = \frac{1}{2}$, and in the triangle on the right, $\sin 30^\circ = \frac{\text{opp}}{\text{hyp}} = \frac{2}{4} = \frac{1}{2}$. Because all right triangles with a 30° angle are similar, the ratios between side lengths are always the same. Therefore, $\sin 30^\circ$ is always equal to $\frac{1}{2}$.

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In trigonometry, angles are often expressed in radians. For example, $\sin\frac{\pi}{6}=\frac{1}{2}$, where $\frac{\pi}{6}$ radians is $30°$. On the SAT, you should assume that any trigonometric expression involving π is in radians, not degrees.

Many students over-complicate trigonometry because they view $\sin x$, $\cos x$, and $\tan x$ as different from regular numbers. Perhaps because of the notation, students sometimes make mistakes like the following:

$$\frac{\sin 2x}{x}=\sin 2$$

The operation above is not possible because $\sin 2x$ is one "entity." You cannot separate sin and $2x$ and treat them independently just like you can't mathematically separate $f(x)$ into f and x.

The definitions of sine, cosine, and tangent are best memorized through the acronym **SOH-CAH-TOA**, S for sine (opposite over hypotenuse), C for cosine (adjacent over hypotenuse), and T for tangent (opposite over adjacent).

Aside from the definitions, you should also memorize the following relationship between the sine and cosine of complementary angles (angles whose measures sum to $90°$):

The sine of an angle is equal to the cosine of the complementary angle.

$$\sin x^\circ=\cos(90^\circ-x^\circ)$$

The reverse is true. The cosine of an angle is equal to the sine of the complementary angle.

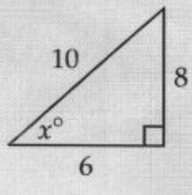
$$\cos x^\circ=\sin(90^\circ-x^\circ)$$

Expressed in radians,

$$\sin\theta=\cos\left(\frac{\pi}{2}-\theta\right)\quad\text{and}\quad\cos\theta=\sin\left(\frac{\pi}{2}-\theta\right)$$

where θ is the measure of the angle in radians.

EXAMPLE 1:



In the triangle shown, what is the value of $\cos x^\circ$?

A) $\frac{3}{5}$ B) $\frac{3}{4}$ C) $\frac{4}{5}$ D) $\frac{4}{3}$

$$\cos x^\circ=\frac{\text{adjacent}}{\text{hypotenuse}}=\frac{6}{10}=\frac{3}{5}$$

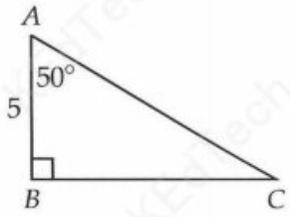
Answer $\boxed{(A)}$.

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EXAMPLE 2: In triangle ABC, angle B is a right angle, angle A has a measure of $50°$, and $AB = 5$. Which of the following represents the length of $\overline{BC}$?

A) $5\cos 50°$ B) $\dfrac{5}{\cos 50°}$ C) $5\tan 50°$ D) $\dfrac{5}{\tan 50°}$

This is the classic trigonometry problem that illustrates the basic purpose of sine, cosine, and tangent: to help us find certain side lengths of a right triangle.



We're given the measure of angle A. The side we're given, $\overline{AB}$, is adjacent to angle A, and the side we're looking for, $\overline{BC}$, is opposite angle A. Now which of the three trig functions relates the opposite and adjacent sides? Tangent. So let's construct an equation using $\tan(A)$:

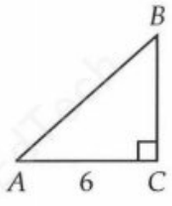
$$\tan(A) = \frac{\text{opp}}{\text{adj}} = \frac{BC}{AB}$$

$$\tan 50° = \frac{BC}{5}$$

$$5\tan 50° = BC$$

Answer $\boxed{(C)}$. Using a calculator to get a better sense of the actual length, we get $BC = 5\tan 50° = 5.96$.

EXAMPLE 3: In triangle ABC, angle C is a right angle, $AC = 6$, and $\cos(A) = \dfrac{3}{4}$. The length of $\overline{BC}$ can be written as $2\sqrt{k}$, where k is a constant. What is the value of k?



Since $\cos(A) = \dfrac{\text{adj}}{\text{hyp}} = \dfrac{AC}{AB}$,

$$\frac{AC}{AB} = \frac{3}{4}$$

Cross-multiplying yields

$$3AB = 4AC$$

$$AB = \frac{4}{3}AC = \frac{4}{3}(6) = 8$$

By the pythagorean theorem, $BC = \sqrt{AB^2 - AC^2} = \sqrt{8^2 - 6^2} = \sqrt{28} = 2\sqrt{7}$. Therefore, $k = \boxed{7}$.

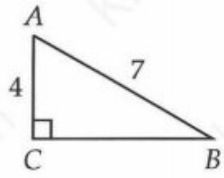
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EXAMPLE 4: In triangle ABC, $\sin(B) = \frac{4}{7}$ and angle C is a right angle. What is the value of $\cos(A)$?

A) $\frac{\sqrt{33}}{7}$ B) $\frac{4}{7}$ C) $\frac{33}{49}$ D) $\frac{16}{49}$

Solution 1: Since C is the right angle in the triangle, the measures of angles A and B must sum to $90°$. Since the cosine of an angle is equal to the sine of the complementary angle, $\cos(A) = \sin(90 - A) = \sin(B) = \frac{4}{7}$. Essentially, $\cos(A)$ is equal to $\sin(B)$, so the value of $\cos(A)$ is also $\frac{4}{7}$. Answer $\boxed{(B)}$.

Solution 2: Since $\sin(B) = \frac{4}{7}$, we can let AC, the side opposite angle B, have a length of 4 and AB, the hypotenuse, have a length of 7.



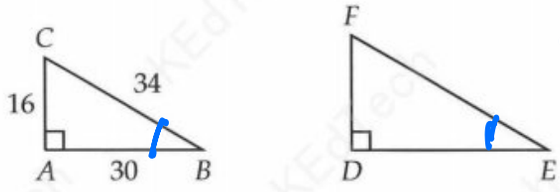
Note that these may not be the actual side lengths of triangle ABC—we're just using convenient numbers to help us derive the value of $\cos(A)$, and we're able to do so because the actual side lengths, whatever they are, must be in the same ratio. With these lengths in place, we get $\cos(A) = \frac{\text{adj}}{\text{hyp}} = \frac{AC}{AB} = \frac{4}{7}$. Answer $\boxed{(B)}$. Note that $\tan(A)$ can also be found by first using the pythagorean theorem to find BC.

EXAMPLE 5:

$$AB = 30$$
$$BC = 34$$
$$AC = 16$$

The side lengths of right triangle ABC are given. Triangle ABC is similar to triangle DEF, where A corresponds to D and B corresponds to E. What is the value of $\sin(E)$?

Since BC is the longest side, it must be the hypotenuse. Angle A is then the right angle.



$\sin B = \frac{16}{34} = \frac{8}{17}$

$= \sin(E)$

Since angle E corresponds to angle B, they must have equal measures. Thus,

$$\sin(E) = \sin(B) = \frac{AC}{BC} = \frac{16}{34} = \boxed{\frac{8}{17}}$$

As we learned at the beginning of this chapter, the sine of an angle in one right triangle is equal to the sine of the corresponding angle in a similar triangle because the ratios between side lengths are the same across similar triangles. This, of course, is true not just for sine but cosine and tangent as well.

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