13 Systems of Equations

A system of equations refers to 2 or more equations that deal with the same set of variables.

$$-5x + y = -7$$
$$-3x - 2y = -12$$

There are two main ways of solving systems of 2 equations: substitution and elimination.

Substitution

Substitution is all about isolating one variable, either x or y, in the fastest way possible.

Taking the example above, we can see that it's easiest to isolate y in the first equation because it has no coefficient. Adding $5x$ to both sides, we get

$$y = 5x - 7$$

We can now substitute the y in the second equation with $5x - 7$ and solve from there.

$$\begin{aligned} -3x - 2(5x - 7) &= -12 \\ -3x - 10x + 14 &= -12 \\ -13x &= -26 \\ x &= 2 \end{aligned}$$

Plugging $x = 2$ back into $y = 5x - 7$, we get $y = 5(2) - 7 = 3$.

The solution is then $x = 2, y = 3$, which can be denoted as $(2, 3)$.

Elimination

Elimination is about getting the same coefficient for one variable across the two equations so that adding or subtracting the equations eliminates that variable.

Using the same example as before, we can multiply the first equation by 2 so that the y's have the same coefficient (we don't have to worry about the sign because we can add or subtract the equations).

$$-10x + 2y = -14$$
$$-3x - 2y = -12$$

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To eliminate y, we add the equations.

$$\begin{aligned} -10x + 2y &= -14 \\ -3x - 2y &= -12 \\ \hline -13x &= -26 \end{aligned}$$

Now, we can see that $x = 2$. This result can be used in either of the original equations to solve for y. Let's pick the first equation.

$$\begin{aligned} -10(2) + 2y &= -14 \\ -20 + 2y &= -14 \\ 2y &= 6 \\ y &= 3 \end{aligned}$$

And finally, we get the same solution as we got using substitution: $x = 2, y = 3$.

When solving systems of equations, you can use either method, but one of them will typically be faster. If you see a variable with no coefficient, like in $-5x + y = -7$ above, substitution is likely the best route. If you see matching coefficients, or you see that it's easy to get matching coefficients, elimination is likely the best route. The example above was simple enough for both methods to work well (though substitution was slightly faster). In these cases, it comes down to your personal preference.

Infinitely many solutions

In the *Other Types of Equation Questions* chapter, we learned that a single equation has infinitely many solutions when both sides of the equation are equivalent.

Similarly, a system of equations has infinitely many solutions when both equations are equivalent. Here, we have to clarify what we mean by "equivalent." Both equations in the system

$$\begin{aligned} 3x + 2y &= 5 \\ 3x + 2y &= 5 \end{aligned}$$

have exactly the same coefficients and constants, and therefore the system has infinitely many solutions ($(1, 1), (3, -2), (5, -5)$, to name just a few). However, the equations in the system

$$\begin{aligned} 6x + 4y &= 10 \\ 3x + 2y &= 5 \end{aligned}$$

do not have the same coefficients and constants, yet the system still has infinitely many solutions. Why is that? Because the two equations are *essentially* the same: the first equation can be divided by 2 to obtain the second equation. In other words, the two equations are equivalent.

Two equations are equivalent when one equation can be manipulated to form the other. A system has infinitely many solutions when the equations in the system are equivalent.

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EXAMPLE 1:

$$3x - 5y = 8$$
$$mx - ny = 32$$

In the given system of equations, m and n are constants. The system has infinitely many solutions. What is the value of $m + n$?

For there to be infinitely many solutions, both equations need to be equivalent. We can multiply the first equation by 4 to get the constants to match:

$$12x - 20y = 32$$
$$mx - ny = 32$$

Now we can clearly see that $m = 12$ and $n = 20$. Therefore, $m + n = \boxed{32}$.

No solutions

In the *Other Types of Equation Questions* chapter, we saw that a single equation has no solutions when both sides of the equation are equivalent except for the constants.

Similarly, a system of equations has no solutions when both equations are equivalent except for their constants. The constants must be different. For example, the system

$$3x + 2y = 5$$
$$3x + 2y = -4$$

has no solutions because the different constants (5 vs. -4) result in equations that contradict each other. There isn't an x and a y that can possibly satisfy both equations at the same time. Note that the system

$$3x + 2y = 5$$
$$6x + 4y = -8$$

also has no solution. Why? Because the second equation can be divided by 2 to get the same system as before.

A system has no solution when the equations are equivalent except for their constants—in other words, when one equation can be manipulated to have the same coefficients as the other. The constants need not match and must not match.

EXAMPLE 2:

$$-ax - 12y = 15$$
$$4x + 3y = -2$$

In the given system of equations, a is a constant. The system has no solution. What is the value of a ?

We must get the coefficients of y to match so that we can compare the two equations. To do that, we can multiply the second equation by -4:

$$-ax - 12y = 15$$
$$-16x - 12y = 8$$

Now that the coefficients of y are both -12, it's easy to see that when $a = \boxed{16}$, the equations are equivalent except for the constants, which must be the case for a system with no solution.

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Word problems

You will most definitely run into a question that asks you to translate a situation into a system of equations. Here's a classic example:

EXAMPLE 3: A group of 30 students order lunch from a restaurant. Each student gets either a burger or a salad. The price of a burger is \$5 and the price of a salad is \$6. If the group spent a total of \$162, how many students ordered burgers?

Let x be the number of students who ordered burgers and y be the number who ordered salads. Based on the given information, we can construct a system of two equations:

$$\begin{aligned} x+y &= 30 \\ 5x+6y &= 162 \end{aligned}$$

Make sure you understand how we came up with these equations. To solve this system, we'll use elimination. Multiply the first equation by 6 and subtract:

$$\begin{aligned} 6x+6y &= 180 \\ 5x+6y &= 162 \\ \hline x &= \boxed{18} \end{aligned}$$

18 students got burgers.

Other Systems of Equations Questions

EXAMPLE 4:

$$\begin{aligned} 3a+6b &= 13 \\ -2a-7b &= 7 \end{aligned}$$

The solution to the given system of equations is (a,b). What is the value of $a-b$?

In this example, the question does not ask for the value of a or b, but the value of the expression $a-b$. In these cases, you can often get directly to the answer by adding or subtracting the equations, rather than solving for each variable individually. Here, we can add the two equations to get $a-b=\boxed{20}$.

EXAMPLE 5:

$$\begin{aligned} 0.05x+0.8 &= 4 \\ 3(0.05x+0.8)+5(3y-0.78) &= 20 \end{aligned}$$

The solution to the given system of equations is (x,y). What is the value of $3y-0.78$?

We can solve for $3y-0.78$ as if it were a single variable or unit. Also, because $0.05x+0.8$ shows up in both equations, we can substitute its value given by the first equation into the second:

$$\begin{aligned} 3(0.05x+0.8)+5(3y-0.78) &= 20 \\ 3(4)+5(3y-0.78) &= 20 \\ 5(3y-0.78) &= 8 \\ 3y-0.78 &= \boxed{1.6} \end{aligned}$$

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You might encounter systems of equations that are a bit more complicated than the standard ones we've already shown. For these systems, substitution and some equation manipulation will typically do the trick.

EXAMPLE 6:

$$y + 3x = 0$$
$$x^2 + 2y^2 = 76$$

The solution to the given system of equations is (x, y). What is a possible value of y ?

In the first equation, we isolate y to get $y = -3x$. Plugging this into the second equation,

$$x^2 + 2(-3x)^2 = 76$$
$$x^2 + 2(9x^2) = 76$$
$$x^2 + 18x^2 = 76$$
$$19x^2 = 76$$
$$x^2 = 4$$
$$x = \pm 2$$

When $x = 2$, $y = -3(2) = -6$. When $x = -2$, $y = -3(-2) = 6$. A possible value of y is then either $\boxed{-6}$ or $\boxed{6}$.

Graphs

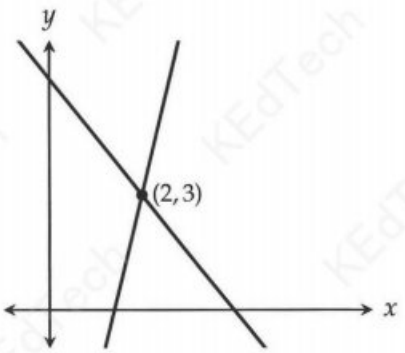
Graphing systems of equations can inform our understanding of them.

More specifically, **when the equations in a system are graphed, the intersection points are the solutions to that system.** Therefore, the number of solutions to a system of equations is equal to the number of intersection points.

Take, for example, the system of equations at the beginning of this chapter:

$$-5x + y = -7$$
$$-3x - 2y = -12$$

We can transform both equations into $y = mx + b$ form (we won't show that here) and graph them to get the following lines.



The solution to the system, $(2, 3)$, is the intersection point. There is only one intersection point, so there is only one solution.

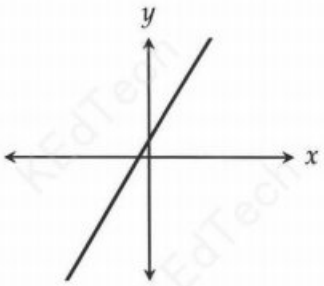
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What about the graphs of systems that have infinitely many solutions or no solution?

Graphing the following system,

$$2y - 4x = 2$$
$$y - 2x = 1$$

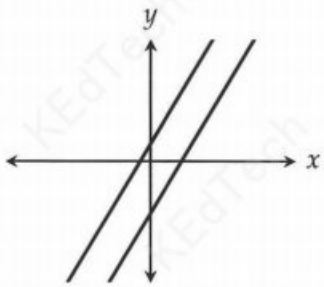
which has infinitely many solutions because its equations are equivalent, we get



It's just one line! Well, actually it's two lines, but because they're the same line, they overlap and intersect at an infinite number of points. Hence, an infinite number of solutions.

Now let's graph a system with no solution:

$$y - 2x = 1$$
$$y - 2x = -3$$



What do you notice about the lines? They're parallel! They have no intersection points. Hence, no solution.

EXAMPLE 7: In the xy-plane, the graphs of the equations $y = 3x - 5$ and $y = -2x + 10$ intersect at the point (h, k). What is the value of k ?

As mentioned earlier, the solutions to a system of equations are the intersection points of the graphs of those equations, and vice versa. **So to find the point(s) where two graphs intersect, solve the system consisting of their equations.** In this problem, that system is

$$y = 3x - 5$$
$$y = -2x + 10$$

Substituting the first equation into the second, we get

$$3x - 5 = -2x + 10$$
$$5x = 15$$
$$x = 3$$

When $x = 3$, $y = 3(3) - 5 = 4$. So the two lines intersect at $(3, 4)$ and $k = \boxed{4}$.

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EXAMPLE 8:

$$y = x^2 - 5x + 6$$
$$y = x + 1$$

The graphs of the equations in the given system of equations intersect at the point (x, y) in the xy-plane. What is a possible value of y ?

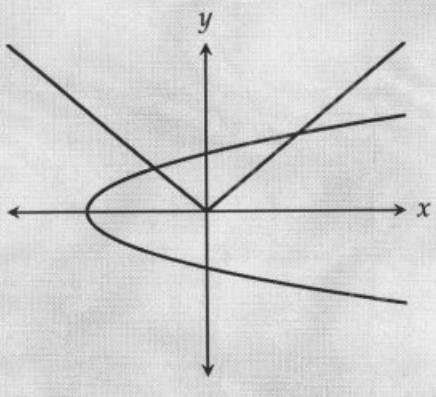
The solutions to the system are the intersection points, so let's solve the system. Substituting the first equation into the second, we get

$$x^2 - 5x + 6 = x + 1$$
$$x^2 - 6x + 5 = 0$$
$$(x-1)(x-5) = 0$$
$$x = 1 \text{ or } 5$$

When $x = 1$, $y = 1 + 1 = 2$. When $x = 5$, $y = 5 + 1 = 6$. So the graphs of the two equations intersect at $(1, 2)$ and $(5, 6)$, which means the possible values of y are $\boxed{2}$ and $\boxed{6}$.

EXAMPLE 9:

$$y^2 = x + 3$$
$$y = |x|$$



The given system of equations is represented in the graph. How many solutions does the system have?

A) Zero B) Exactly one C) Exactly two D) Infinitely many

Simple. The graphs intersect at two points, so the system has two solutions. Answer $\boxed{(C)}$.

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EXERCISE 1: Answers for this exercise start on page 360.

1

$$y = -5x + 65$$
$$y = 25$$

Which ordered pair (x, y) is a solution to the given system of equations?

A) $(-8, 25)$
B) $(8, 25)$
C) $(25, -8)$
D) $(25, 8)$

2

$$3x - 5y = -11$$
$$x = 1 - 3y$$

The solution to the given system of equations is (x, y). What is the value of y ?

A) -1
B) 0
C) 1
D) 2

3

$$y + 2x = 20$$
$$6x - 5y = 12$$

What is the solution (x, y) to the given system of equations?

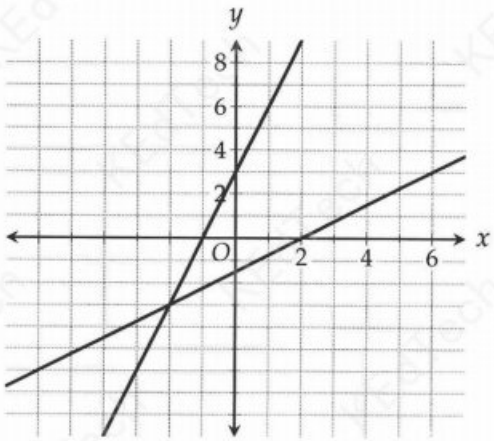
A) $(-7, 6)$
B) $(-6, 7)$
C) $(6, 7)$
D) $(7, 6)$

4

$$y = x^2 + 10x + 23$$
$$x - 9 = -15$$

The solution to the given system of equations is (x, y). What is the value of y ?

5



The graph of a system of two linear equations is shown. What is the solution (x, y) to the system?

A) $(-4, -6)$
B) $(-3, -2)$
C) $(-2, -3)$
D) $(-1, 2)$

6

$$2x + 5y = 24$$
$$x + 4y = 15$$

The solution to the given system of equations is (x, y). What is the value of $x + y$?

A) 7
B) 8
C) 9
D) 10

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7

$$2x - 4y = 8$$
$$x + 2y = 4$$

How many solutions (x, y) does the given system of equations have?

A) Zero
B) One
C) Two
D) More than two

8

An ice cream vendor sells small and large cups of ice cream. Each small cup of ice cream is sold for \$5, and each large cup is sold for \$7. During an event, the vendor sold a total of 200 cups of ice cream and collected \$160 more from sales of small cups than from sales of large cups. If x is the number of small cups sold at the event and y is the number of large cups sold at the event, which system of equations represents this situation?

A) $x - y = 160$
$5x + 7y = 200$

B) $x - y = 160$
$7x + 5y = 200$

C) $x + y = 200$
$5x = 7y + 160$

D) $x + y = 200$
$7y = 5x + 160$

9

If $\frac{a}{b} = 4.5$ and $\frac{a}{kb} = 12.5$, what is the value of k?

10

$$6x - 3y = 9$$

One of the two linear equations in a system is given. The system has no solution. Which equation could be the second equation in the system?

A) $3x - y = 2$
B) $2x - y = 3$
C) $\frac{1}{3}x - \frac{1}{6}y = 2$
D) $\frac{1}{3}x - \frac{1}{6}y = \frac{1}{2}$

11

$$x^2 - y^2 = 48$$
$$x - y = 6$$

If (x, y) is the solution to the given system of equations, what is the value of $x + y$?

A) 8
B) 12
C) 16
D) 24

12

$$2x - 5y = a$$
$$bx + 10y = -8$$

In the given system of equations, a and b are constants. If the system has infinitely many solutions, what is the value of a?

A) -4
B) $\frac{1}{4}$
C) 4
D) 16

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13

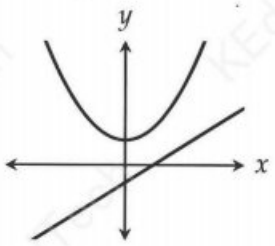
$$y = ax + b$$
$$y = -bx$$

In the given system of equations, a and b are constants. The graphs of the equations in the xy-plane intersect at the point $(2, 8)$. What is the value of a ?

A) 2
B) 4
C) 6
D) 8

14

$$y = x^2 + 1$$
$$y = x - 1$$



The given system of equations is represented in the graph. How many solutions does the system have?

A) Zero
B) Exactly one
C) Exactly two
D) Exactly three

15

$$3x - \frac{5}{16} = 4$$
$$y = 2\left(3x - \frac{5}{16}\right)^2$$

The solution to the given system of equations is the ordered pair (x, y). What is the value of y ?

A) 8
B) 16
C) 32
D) 64

16

$$ax + 2y = 5$$
$$3x - 6y = 20$$

In the given system of equations, a is a constant. The system has exactly one solution. Which of the following could NOT be the value of a ?

A) -1
B) $\frac{3}{4}$
C) 1
D) 3

17

$$3x = 16$$
$$80 = 4y^2 + 3x$$

The graphs of the equations in the given system of equations intersect at the point (x, y) in the xy-plane. What is a possible value of y ?

A) $-\frac{16}{3}$
B) -4
C) $\frac{4}{3}$
D) 8

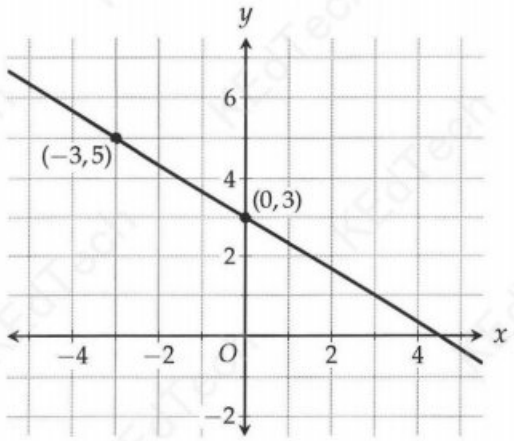
18

$$-5x = 8 - ky$$
$$18y + 5x = -10x + 21$$

In the given system of equations, k is a constant. If the system has no solution, what is the value of k ?

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19



The graph of one of the two equations in a linear system is shown in the xy-plane. The system has no solution. Which equation could be the second equation in this system?

A) $y = -\frac{3}{2}x - 1$

B) $y = \frac{3}{2}x + 3$

C) $y = -\frac{2}{3}x - 1$

D) $y = -\frac{2}{3}x + 3$

20

In the xy-plane, the graph of $y = x^2 - 7x + 7$ intersects the graph of $y = 2x - 1$ at the points $(1, 1)$ and (p, q). What is the value of p ?

21

Shannon drove 340 miles from Melrose to Norwich, passing Hanover along the way. From Melrose to Hanover, her average speed was 50 miles per hour, and from Hanover to Norwich, her average speed was 30 miles per hour. The drive took 9 hours. How many miles did Shannon drive from Melrose to Hanover?

22

$$\frac{4}{3} + \frac{5}{4}x = 28y + \frac{5}{8}x$$

$$my = \frac{1}{2}(5x - 8)$$

In the given system of equations, m is a constant. If the system has no solution, what is the value of m ?

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