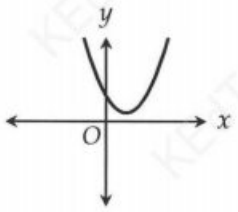
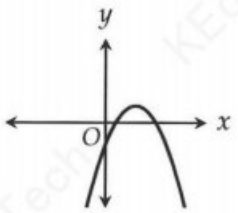
# 16 Quadratic Functions

In a previous chapter, we learned how to solve quadratic equations. In this chapter, we'll discuss quadratic functions and their graphs, called **parabolas**. A parabola is a "U" shape that opens either upward or downward.



upward-opening



downward-opening

We can determine whether a parabola opens upward or downward without graphing it just by looking at the standard form of its equation.

The equation of a quadratic function in **standard form** is

$$y = ax^2 + bx + c$$

where $a$, $b$, and $c$ are constants. The coefficient of $x^2$, $a$, determines whether the graph (parabola) opens upward or downward.

- When $a$ is positive, the parabola opens upward.
- When $a$ is negative, the parabola opens downward.

The constant $c$ is the $y$-intercept of the graph. In other words, the graph passes through $(0, c)$.

Let's walk through an example quadratic function to demonstrate.

**EXAMPLE QUADRATIC 1:**

$$y = x^2 - 4x - 21$$

This example equation is written in standard form, where $a = 1$, $b = -4$, and $c = -21$. Because $a$ is positive, the graph of this equation opens upward. And because $c = -21$, the graph intercepts the $y$-axis at $(0, -21)$.



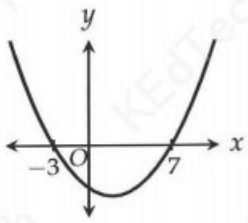
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Let's keep going with this example.

We already know the $y$-intercept, so let's find the $x$-intercepts by using what we learned in the quadratic equations chapter. The $x$-intercepts are the points at which a graph crosses the $x$-axis. To find them, set $y$ equal to 0 and solve for $x$. In this case, setting $y$ equal to 0 gives us a plain old quadratic equation we already know how to solve, by factoring or by applying the quadratic formula. Here's the factoring approach:

$$\begin{aligned} y &= x^2 - 4x - 21 \\ 0 &= (x+3)(x-7) \\ x &= -3, 7 \end{aligned}$$

Therefore, the graph of Example Quadratic 1 has $x$-intercepts at $(-3, 0)$ and $(7, 0)$, as shown below.




The $x$-coordinates of the $x$-intercepts are sometimes referred to as the **roots** of a quadratic, the values of $x$ for which $y = 0$. In this example, the roots are $-3$ and 7.

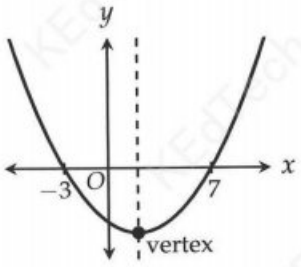
The $x$-intercepts of a quadratic function are obvious when its equation is written in **intercept form**:

$$y = a(x-p)(x-q)$$

where $p$ and $q$ are the $x$-coordinates of the $x$-intercepts (the roots), and $a$ is a constant equal to the value of $a$ from the standard form equation of the quadratic ($y = ax^2 + bx + c$).

Note that all the formulas we learned in the quadratic equations chapter apply to this chapter as well. The sum of the roots, $-3 + 7 = 4$, is equivalent to $-\frac{b}{a}$ ($-\frac{-4}{1} = 4$), and the product of the roots, $-3 \times 7 = -21$, is equivalent to $\frac{c}{a}$ ($\frac{-21}{1} = -21$). These formulas allow us to obtain the sum or product of the roots without having to solve for the roots themselves.

Now let's take the graph above and draw a line down the middle of the parabola such that each side is a mirror of the other.



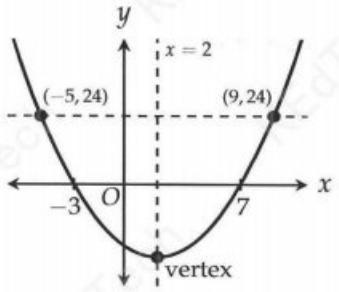

This line is called the **axis of symmetry** and always passes through a parabola's **vertex**. You can think of the vertex as the "midpoint" of a parabola. For parabolas that open upward, the vertex is the point where they reach their minimum (their lowest point). For parabolas that open downward, the vertex is the point where they reach their maximum (their highest point).

Now how do we find the coordinates of the vertex? Let's start by finding the equation of the axis of symmetry.



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If we draw a horizontal line across our example parabola, the line will intersect the parabola at two points.




For the line we've drawn above, the intersection points are $(-5, 24)$ and $(9, 24)$.

Notice that the axis of symmetry runs through the middle of these two intersection points. This would be true for any horizontal line we could have drawn.

Because the axis of symmetry lies in the middle, we can find its location by computing the midpoint of the $x$-coordinates of the two intersection points. The midpoint of any two values is just their average, so the midpoint of $-5$ and 9 is $\frac{-5+9}{2} = 2$. The equation of the axis of symmetry must then be $x = 2$.

Since the vertex lies on the axis of symmetry, it must have an $x$-coordinate of 2. To find the $y$-coordinate of the vertex, we plug $x = 2$ into the equation of the parabola (Example Quadratic 1):

$$y = (2)^2 - 4(2) - 21 = -25$$

Therefore, the vertex is at $(2, -25)$, and this parabola reaches a minimum value of $-25$.

Now you might be thinking that we were only able to find the vertex this way because we plucked those intersection points out of nowhere. Indeed, finding the vertex would be very tedious if, for every parabola, we had to draw a horizontal line and solve for the intersection points ourselves.

Thankfully, there's an easier way. Look back at the graph above, and notice that the $x$-axis itself is a horizontal line—one for which we already knew the intersection points: $(-3, 0)$ and $(7, 0)$, the $x$-intercepts! If we take the average of their $x$-coordinates, $-3$ and 7, we get $x = \frac{-3+7}{2} = 2$, the same value we got earlier.

The conclusion here is that for any given parabola, the $x$-coordinate of its vertex is simply the midpoint, or average, of its roots.

- Given any two distinct points on a parabola that have the same $y$-value, say $(x_1, y)$ and $(x_2, y)$, the $x$-coordinate of the vertex is the average of the two $x$-coordinates, $\frac{x_1 + x_2}{2}$.
- By extension, the $x$-coordinate of the vertex is equal to the average of the roots.
- For a parabola whose equation is in standard form, $y = ax^2 + bx + c$, the sum of the roots is $-\frac{b}{a}$. Since the average of the roots is the sum of the roots divided by 2, the $x$-coordinate of the vertex is equal to $-\frac{b}{2a}$.

Let's confirm that the formula $-\frac{b}{2a}$ works for locating the vertex. The parabola above, which has a vertex at $(2, -25)$, is defined by $y = x^2 - 4x - 21$ (Example Quadratic 1). For this equation, the value of $-\frac{b}{2a}$ is $-\frac{-4}{2(1)} = 2$, which is indeed the $x$-coordinate of the vertex.


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Now that we know how to find the vertex of a parabola, let's talk about vertex form.

The equation of a quadratic function in **vertex form** is

$$y = a(x-h)^2 + k$$

where $(h,k)$ is the vertex, and $a$ is a constant equal to the value of $a$ from the standard form equation of the quadratic ($y = ax^2 + bx + c$).

Let's express Example Quadratic 1 in vertex form. Its equation in standard form is $y = x^2 - 4x - 21$ and its vertex, which we already found, is $(2, -25)$. The equation of this quadratic in vertex form is then

$$y = (x-2)^2 - 25$$

Because the coefficient of $x^2$ in standard form is 1, the value of $a$ in vertex form is also 1. Note that if we expand this result, we get back the original standard form:

$$y = (x-2)^2 - 25 = (x^2 - 4x + 4) - 25 = x^2 - 4x - 21$$

Vertex form is called vertex form for a reason—we're able to find the vertex just by looking at the numbers in the equation. Here are some quick examples:

- The graph of $y = -4(x-3)^2 - 16$ has a vertex at $(3, -16)$.
- The graph of $y = 3(x-4)^2 + 10$ has a vertex at $(4, 10)$.
- The graph of $y = -2(x+5)^2 + 8$ has a vertex at $(-5, 8)$.

One of the most common mistakes students make is to look at an equation like $y = -2(x+5)^2 + 8$ and think the vertex is at $(5,8)$ instead of $(-5,8)$. The routine I use to avoid this mistake is to ask, *What value of $x$ would make the expression inside the parentheses equal to zero? Well, $x = -5$ would make $x + 5$ equal to 0. Therefore, the vertex is at $x = -5$.*

**The Discriminant**

Let's say that we have a quadratic function expressed in standard form, $y = ax^2 + bx + c$, and we have to find the $x$-intercepts of its graph. To do so, we set $y$ equal to 0 and solve the resulting quadratic equation:

$$0 = ax^2 + bx + c$$

A generalized solution is given by the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Notice the expression tucked underneath the square root. This expression, $b^2 - 4ac$, is called the discriminant.

- When $b^2 - 4ac > 0$, the plus or minus in the quadratic formula yields two distinct real solutions to the equation. Thus, the graph of the quadratic function intersects the $x$-axis twice (two $x$-intercepts).

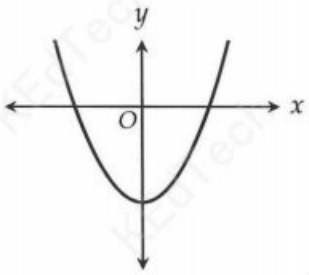


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- When $b^2 - 4ac = 0$, a zero ends up underneath the square root, and the square root of zero is zero. Since there is no difference between plus zero and minus zero, the quadratic formula yields only one distinct real solution to the equation. Thus, the graph of the quadratic function intersects the $x$-axis only once (one $x$-intercept).
- When $b^2 - 4ac < 0$, a negative number ends up underneath the square root. Since the square root of a negative number cannot be found, the quadratic formula does not yield any real solutions to the equation. Thus, the graph of the quadratic function does not intersect the $x$-axis at all (zero $x$-intercepts).

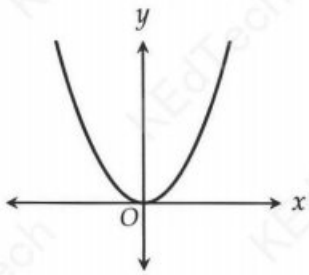
The value of the discriminant does not matter. What matters is whether it's positive, negative, or zero. Here's another summary of the three possibilities, with $D$ representing the value of the discriminant:

When $D > 0$,



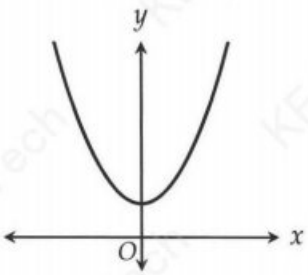

two roots
(two real solutions)

When $D = 0$,




one root
(one real solution)

When $D < 0$,




no roots
(no real solutions)

Let's calculate the discriminant for Example Quadratic 1, $y = x^2 - 4x - 21$,

$$\text{Discriminant} = b^2 - 4ac = (-4)^2 - 4(1)(-21) = 100$$

Because the discriminant is positive, the graph must have two $x$-intercepts. And indeed it does: $(-3, 0)$ and $(7, 0)$, the two $x$-intercepts we found earlier.

---

Now that we've illustrated all the properties of quadratics you need to know, let's reinforce the concepts by going through one more example quadratic.

**EXAMPLE QUADRATIC 2:**

$$y = -2x^2 - 12x - 25$$

- Again, we're given an equation in standard form. Because the coefficient of $x^2$, $-2$, is negative, the graph of this quadratic opens downward.
- The constant term, $-25$, indicates that the graph has a $y$-intercept at $(0, -25)$.
- The discriminant is equal to $b^2 - 4ac = (-12)^2 - 4(-2)(-25) = -56$. Because the discriminant is negative, the graph does not intersect the $x$-axis (there are no $x$-intercepts or roots).
- The sum of the roots is $-\frac{b}{a} = -\frac{-12}{-2} = -6$, and the product of the roots is $\frac{c}{a} = \frac{-25}{-2} = 12.5$. But wait! We just determined that there were no roots, so how can there be a sum and a product of roots that don't exist? The answer has to do with imaginary numbers. This quadratic doesn't have any *real* roots, but it does have *imaginary* roots, and their sum and product are equal to the values we just computed. On the SAT, you won't be tested on imaginary numbers—it's good to be aware of them, but ultimately they're not something you need to worry about.



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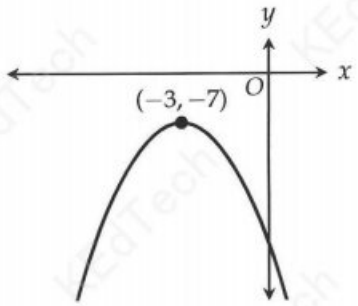
- The vertex of the graph is located at $x = -\frac{b}{2a} = -\frac{-12}{2(-2)} = -3$. Note that this formula works for any quadratic function, even ones that don't have any real roots. Now we can plug $x = -3$ into the equation to find the vertex's $y$-coordinate: $y = -2(-3)^2 - 12(-3) - 25 = -7$. Therefore, the vertex is at $(-3, -7)$.
- Now that we've found the vertex, we can express Example Quadratic 2 in vertex form:

$$y = -2(x+3)^2 - 7$$

The $-2$ comes from the coefficient of $x^2$ given by the standard form equation. Note that a process called *completing the square* can also be used to convert from standard form to vertex form. Knowing how to complete the square is not necessary for the quadratics questions you'll encounter on the SAT, but it is necessary for other question types. We'll be covering how to complete the square in a later chapter.

- Because the graph opens downward, it reaches a maximum at its vertex. Thus, the graph reaches a maximum value of $-7$ at $x = -3$.

Finally, a graph of Example Quadratic 2:




Wow! We just covered pretty much everything you need to know about quadratics. We're not quite done yet though as there are several unique question types you need to be exposed to.

**EXAMPLE 1:**

$$f(x) = -2(x-1)(x-13)$$

The function $f$ is defined by the given equation. For what value of $x$ does $f(x)$ reach its maximum?

**Solution 1:** Since the graph of $f$ is a parabola that opens downward, $f(x)$ reaches its maximum at its vertex. Now how do we locate the vertex? Well, $f$ has two roots, $x = 1$ and $x = 13$. The $x$-coordinate of the vertex is equal to the average of these two roots: $\frac{1+13}{2} = 7$. Therefore, $f(x)$ reaches its maximum at $x = \boxed{7}$.

**Solution 2:** We can expand $f(x)$ like so,

$$f(x) = -2(x^2 - 14x + 13) = -2x^2 + 28x - 26$$

and use the following formula to find the $x$-coordinate of the vertex:

$$x = -\frac{b}{2a} = -\frac{28}{2(-2)} = \boxed{7}$$



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**EXAMPLE 2:**

$$-3x^2 + 10x - 7 = 0$$

How many distinct real solutions does the given equation have?

A) Exactly one B) Exactly two C) Zero D) Infinitely many

The discriminant, $b^2 - 4ac$, tells us how many solutions a quadratic equation has. In this case,

$$b^2 - 4ac = (10)^2 - 4(-3)(-7) = 100 - 84 = 16$$

The discriminant is positive, so the given equation must have two distinct real solutions. Answer $\boxed{(B)}$.

**EXAMPLE 3:**

$$2x^2 + 12x + c = 0$$

In the given equation, $c$ is an integer constant. If the equation has no real solution, what is the least possible value of $c$ ?

For a quadratic equation to have no real solution, the discriminant must be less than zero:

$$\begin{aligned} b^2 - 4ac &< 0 \\ (12)^2 - 4(2)(c) &< 0 \\ 144 - 8c &< 0 \\ -8c &< -144 \\ c &> 18 \end{aligned}$$

Given that $c$ is an integer constant, the least possible value of $c$ is $\boxed{19}$.

**EXAMPLE 4:**

$$y = x + 5$$
$$y = x^2 + 11x + c$$

In the given system of equations, $c$ is a constant. The system has exactly one distinct real solution. What is the value of $c$ ?

Remember how we solve a system of equations? We use substitution or elimination to get a single equation. Then we solve the single equation for $x$. The solutions we get from the single equation are the solutions to the system.

So for this question, let's start off by substituting the first equation into the second,

$$x + 5 = x^2 + 11x + c$$

and moving all the terms to one side:

$$0 = x^2 + 10x + c - 5$$

We now have a single quadratic equation. Remember—the solutions to this equation are the solutions to the system. **Therefore, for the system to have only one solution, this equation must have only one solution.**



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And what determines how many solutions a quadratic equation has? That's right—the discriminant. For a quadratic equation to have only one real solution, its discriminant must be equal to zero:

$$b^2 - 4ac = 0$$

$$(10)^2 - 4(1)(c-5) = 0$$
$$100 - 4(c-5) = 0$$
$$100 - 4c + 20 = 0$$
$$-4c = -120$$
$$c = \boxed{30}$$

**EXAMPLE 5:** In the $xy$-plane, a parabola with equation $y = x^2 + 5x + 6$ intersects a line at exactly one point. If the line has equation $y = ax - 10$, where $a$ is a positive constant, what is the value of $a$ ?

To find the intersection point(s) of two graphs, we solve the system consisting of their equations. The solutions to the system are the intersection points.

**Therefore, for two graphs to have only one intersection point, the system consisting of their equations must have only one solution.**

This question is essentially just a variation of Example 4. In this case, the system is

$$y = ax - 10$$
$$y = x^2 + 5x + 6$$

Substituting the first equation into the second, we get

$$ax - 10 = x^2 + 5x + 6$$

Moving all the terms to the right-hand side gives the following quadratic equation:

$$0 = x^2 + (5-a)x + 16$$

Notice how we grouped the terms. Now, for the system to have only one solution, this equation must have only one solution. And for that to be true, the discriminant must be equal to zero (like in Example 4):

$$b^2 - 4ac = 0$$

$$(5-a)^2 - 4(1)(16) = 0$$
$$(5-a)^2 = 64$$
$$5 - a = \pm 8$$
$$-a = -5 \pm 8$$
$$-a = -13 \text{ or } 3$$
$$a = 13 \text{ or } -3$$

Since the question specifies that $a$ is a positive constant, $a = \boxed{13}$.



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## Review:

1. The equation of a quadratic function in standard form is
$$y = ax^2 + bx + c$$
where $a$, $b$, and $c$ are constants. The graph of a quadratic function in the $xy$-plane is a parabola.

2. If $a > 0$, the parabola opens upward and has a minimum.
If $a < 0$, the parabola opens downward and has a maximum.

3. The parabola has a $y$-intercept at $(0, c)$.

4. The equation of a quadratic function in intercept form is
$$y = a(x - p)(x - q)$$
where $p$ and $q$ are the $x$-coordinates of the $x$-intercepts (the roots), and $a$ is a constant equal to the value of $a$ from the standard form equation of the quadratic.

5. When the equation of a quadratic is not in intercept form, the $x$-intercepts can be found in the following ways:
   - Factoring (i.e. converting the equation into intercept form)
   - The quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
   - Graphing on Desmos

6. The sum of the roots $= -\dfrac{b}{a}$

7. The product of the roots $= \dfrac{c}{a}$

8. The discriminant, $D = b^2 - 4ac$, can be used to determine how many real solutions a quadratic equation $ax^2 + bx + c = 0$ has.
   - When $D > 0$, the equation has two real solutions (the parabola crosses the $x$-axis twice).
   - When $D = 0$, the equation has one real solution (the parabola crosses the $x$-axis once).
   - When $D < 0$, the equation has no real solutions (the parabola does not cross the $x$-axis at all).

9. The $x$-coordinate of the vertex can be found by
   - Averaging the $x$-coordinates of two points on the parabola that have the same $y$-coordinate (e.g. the $x$-intercepts/roots).
   - Using the formula $x = -\dfrac{b}{2a}$.

   The $x$-coordinate can then be plugged into the equation to find the $y$-coordinate of the vertex.

10. The equation of a quadratic function in vertex form is
$$y = a(x - h)^2 + k$$
where $(h, k)$ is the vertex, and $a$ is a constant equal to the value of $a$ from the standard form equation of the quadratic.

11. The minimum or maximum of a quadratic function is always reached at its vertex.


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**EXERCISE 1:** Answers for this exercise start on page 373.

**1**

In the $xy$-plane, the graph of the equation $y = x^2 - 3x - 10$ has an $x$-intercept at $(5,0)$. What is the other $x$-intercept of the graph?

A) $(-5,0)$
B) $(-2,0)$
C) $(-1,0)$
D) $(2,0)$

**2**

$$E(x) = 50x^2 - 800x + 10{,}000$$

The given equation models the relationship between the total monthly expenses $E$, in dollars, of a restaurant and the number of tables $x$ in its dining area, where $0 \leq x \leq 25$. Which of the following is the best interpretation of 10,000 in this context?

A) The maximum number of tables that can be placed in the dining area
B) The average monthly expenses, in dollars, for each table in the dining room
C) The total monthly expenses, in dollars, when there are zero tables in the dining area
D) The total monthly expenses, in dollars, when the number of tables in the dining area is maximized

**3**

$$v(t) = -17(t - 26)^2 + 14{,}750$$

The given function $v$ models the value, in dollars, of an artifact $t$ years after its excavation. According to this model, how many years after its excavation was the value of the artifact the greatest?

**4**

$$g(x) = -5x^2 - 16x + 4$$

The given equation defines the function $g$. For what value of $x$ does $g(x)$ reach its maximum?

**5**

$$-12x^2 - 6x + d = 0$$

In the given equation, $d$ is a constant. If the equation has exactly one solution, what is the value of $d$ ?

A) $-\frac{1}{4}$
B) $-\frac{2}{3}$
C) $-\frac{3}{4}$
D) $-\frac{3}{2}$

**6**

The function $f(x) = -\frac{1}{2}(x - 5)^2 + 8$ gives the profit, in hundreds of thousands of dollars, a company makes when $x$ thousand units of its product are produced and sold, where $1 \leq x \leq 9$. Which of the following is the best interpretation of the vertex of the graph of $y = f(x)$ in the $xy$-plane?

A) The company can make a maximum profit of 500,000 dollars.
B) The company can make a maximum profit of 800,000 dollars.
C) The company makes a profit of 0 dollars when 5,000 units of its product are produced and sold.
D) The company makes a profit of 0 dollars when 8,000 units of its product are produced and sold.


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**7**

In the $xy$-plane, a parabola has vertex $\left(\frac{7}{3}, -4\right)$ and intersects the $x$-axis at $(10, 0)$ and $(a, 0)$, where $a$ is a constant. What is the value of $a$ ?

A) $-10$

B) $-\frac{23}{3}$

C) $-\frac{19}{3}$

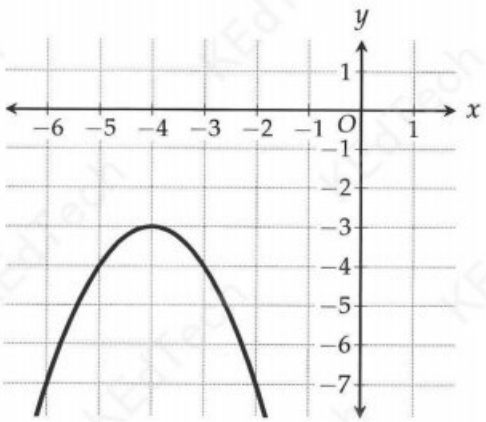
D) $-\frac{16}{3}$

**8**

$$ax^2 + 30x - 9 = 0$$

In the given equation, $a$ is a constant. The equation has no real solutions if $a < k$. What is the greatest possible value of $k$ ?

**9**




What is an equation of the graph shown?

A) $y = -(x-4)^2 - 3$

B) $y = -(x+4)^2 - 3$

C) $y = (x-4)^2 - 3$

D) $y = (x+4)^2 - 3$

**10**

$$9x^2 + 2cx + 36 = 0$$

In the given equation, $c$ is a constant. For which of the following values of $c$ will the equation have no real solution?

A) 26

B) 18

C) $-11$

D) $-19$

**11**

In the $xy$-plane, the graph of the equation $y = a(x-b)^2 - c$, where $a$, $b$, and $c$ are positive constants, could have a vertex at which of the following points?

A) $(7, 10)$

B) $(7, -10)$

C) $(-7, 10)$

D) $(-7, -10)$

**12**

$$f(x) = -4x^2 + 22x$$

The function $f$ gives the download speed, in megabytes per second, of a file $x$ minutes after a download of the file was started. The graph of $y = f(x)$ in the $xy$-plane has $x$-intercepts at $(0, 0)$ and $(b, 0)$, where $b$ is a constant. Which of the following is the best interpretation of $b$ in this context?

A) At the start of the download, the download speed was $b$ megabytes per second.

B) During the download, the download speed reached a maximum of $b$ megabytes per second.

C) The download speed was at its highest $b$ minutes after the download was started.

D) The download was completed $b$ minutes after it was started.



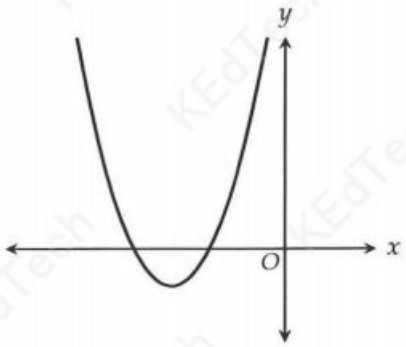
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**13**

$$-x^2 + bx - 961 = 0$$

In the given equation, $b$ is a positive integer. The equation has exactly two distinct real solutions. What is the least possible value of $b$ ?

**14**




The graph of $y = f(x)$ is shown in the $xy$-plane. If $a$, $b$, and $c$ are positive constants, which of the following equations could define the function $f$ ?

A) $f(x) = -a(x + b)(x + c)$
B) $f(x) = -a(x - b)(x - c)$
C) $f(x) = a(x + b)(x + c)$
D) $f(x) = a(x - b)(x - c)$

**15**

In the $xy$-plane, a line with equation $y = x + 3$ intersects a parabola at exactly one point. If the parabola has equation $y = kx^2 - 5x + 7$, where $k$ is a constant, what is the value of $k$ ?

**16**

$$y = 3$$

$$y = x^2 + 6x + r$$

In the given system of equations, $r$ is a constant. For which of the following values of $r$ does the given system have no real solution?

A) $-50$
B) $1$
C) $12$
D) $38$

**17**

$$3y = c$$

$$y = -4x^2 + 10x$$

In the given system of equations, $c$ is a constant. The graphs of the equations in the given system intersect at exactly one point in the $xy$-plane. What is the value of $c$ ?

**18**

$$f(x) = 2x^2 + bx + c$$

The function $f$ is defined by the given equation, where $b$ and $c$ are constants. The graph of $y = f(x)$ in the $xy$-plane is a parabola that has a vertex at $(h, k)$, where $h$ and $k$ are both negative constants. Which of the following must be true?

I. $b > 0$
II. $c \geq 0$

A) I only
B) II only
C) I and II
D) Neither I nor II



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