Prior Knowledge

Math is very much like a staircase; each course is a stair you climb, providing you with the skills and techniques to guide you upward to the next course. This section will give a brief review of important tools that are used to help solve AP Precalculus questions.

Factoring

Factoring is the process of breaking down a polynomial expression or equation and finding its factors that, when multiplied together, return to the original polynomial. A few different factoring techniques will be reviewed.

Greatest Common Factor (GCF)

Greatest common factor factoring refers to dividing out the greatest like term in a polynomial. The greatest common factor can be a number, a variable, or both.

› Example

Factor each of the following polynomials.

1. $3x^4 - x$
2. $2x^3 + 6x^4 - 4x^5$
3. $(x+1)^3 - 4(x+1)^2$

1. $3x^4 - x = x(3x^3 - 1)$

2. $2x^3 + 6x^4 - 4x^5 = 2x^3(1 + 3x - 2x^2)$

3. $(x+1)^3 - 4(x+1)^2 = (x+1)^2((x+1) - 4) = (x+1)^2(x-3)$

$x + 1 - 4 = x - 3$

Difference of Two Squares

Difference of two squares factoring refers to binomials of the form $x^2 - p^2$. To factor a difference of two squares, $x^2 - p^2 = (x-p)(x+p)$. The factors are known as conjugate pairs because each factor has the same terms, x and p, but one factor is a sum and the other factor is a difference.

$(x-p)(x+p) = x^2 + xp - px - p^2 = x^2 - p^2$

K Education & Technology

Example

Factor each of the following polynomials.

1. $16x^2 - 9$
2. $(y-2)^2 - 25$
3. $49 - 4t^2$
4. $x^2 - 5$

1. $16 = 4^2$ $9 = 3^2$ $(4x)^2 - (3)^2 = (4x-3)(4x+3)$

2. $(y-2)^2 - 25 = (y-2)^2 - 5^2 = ((y-2)-5)((y-2)+5)$

$= (y-2-5)(y-2+5) = (y-7)(y+3)$

3. $49 - 4t^2 = (7-2t)(7+2t)$

4. $x^2 - 5 = (x-\sqrt{5})(x+\sqrt{5})$ $5 = (\sqrt{5})^2$

Factoring Trinomials

$(x-p)(x-q) = x^2 - qx - xp + pq = x^2 - (p+q)x + pq$

$x^2 + bx + c$

To factor a quadratic of the form $x^2 + bx + c$, look for two real numbers, p and q, that satisfy $p + q = b$ and $pq = c$.

$b = p + q$ $c = pq$

Example

Factor each of the following polynomials.

1. $y^2 - 24y + 144$
2. $x^2 + 6x + 8$
3. $z^2 - z - 20$

1. $y^2 - 24y + 144 = (y-12)(y-12) = (y-12)^2$

2. $x^2 + 6x + 8 = (x+2)(x+4)$

3. $z^2 - z - 20 = (z-5)(z+4)$

1. x^2

K Education & Technology

AC method ax^2+bx+c

Steps for Factoring Using Split the Middle

1. Find the product *ac*.
2. Determine numbers whose product is *ac* and whose sum is *b*.
3. Split up the middle term *bx*, and write it as a sum of the two numbers determined in step 2.
4. The trinomial is written as an equivalent polynomial with four terms. Group together the first two terms, group together the last two terms, and factor out a common factor from each.
5. A common factor for the resulting two terms should appear that can then be factored out. Doing this will reveal the two factors of the original trinomial.

Example

Factor each of the following quadratics.

1. $25t^2 + 10t + 1$
2. $3a^2 - 10a + 8$

1. $25t^2+10t+1$ $\quad a=25 \quad c=1 \quad ac=25$

$5t \quad 5t$ $\quad 5+5=10=b$

$25t^2+5t+5t+1$

$= 5t(5t+1)+(5t+1) = (5t+1)(5t+1) = (5t+1)^2$

2. $3a^2-10a+8$ $\quad a=3 \quad c=8 \quad ac=24 \quad -6+(-4)=-10=b$

$-6a \quad -4a$

$3a^2-6a-4a+8 = 3a(a-2)-4(a-2)$

$= (3a-4)(a-2)$

K Education & Technology

Sum or Difference of Cubes

To factor a sum or difference of cubes, use the factoring formulas below:

$$x^3 + p^3 = (x + p)(x^2 - px + p^2)$$
$$x^3 - p^3 = (x - p)(x^2 + px + p^2)$$

The trinomial obtained by factoring a sum or difference of cubes does not factor further.

Example

Factor each of the following polynomials.

1. $x^3 - 8$
2. $1 + 1000x^3$

1. $x^3 - 8 = (x - 2)(x^2 + 2x + 4)$

$1 = 1^3$ $\quad 1000x^3 = (10x)^3$ $\quad$ 2. $1 + 1000x^3 = (1 + 10x)(1 - 10x + 100x^2)$

The Quadratic Formula

When asked to solve a quadratic equation, one method is to use factoring and set each factor equal to zero. Sometimes it is impossible to factor a quadratic equation. In those cases, the solutions can be found using the quadratic formula.

The Quadratic Formula

Given a quadratic equation in standard form, $ax^2 + bx + c = 0$, the formula below can find the solutions to the quadratic equation:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example

Solve the following quadratic equations, and write your answer in simplest radical form.

1. $2x^2 - 5x - 7 = 0$
2. $4x^2 = 16x + 20$

$\frac{14}{4} = \frac{7}{2}$ $\quad -1$

$\frac{-16 \pm 2x}{-8}$ $\quad \frac{-16+2x}{-8} = -1$ $\quad \frac{-16-2x}{-8} = 5$

K Education & Technology

2. The quadratic equation needs to be written in standard form first, $-4x^2 + 16x + 20 = 0$. The equation is now in standard form with $a = -4$, $b = 16$, $c = 20$. Substitute the values into the quadratic formula:

$$x = \frac{-(16) \pm \sqrt{(16)^2 - 4(-4)(20)}}{2(-4)}$$

$$x = \frac{-16 \pm \sqrt{256 + 320}}{-8}$$

$$x = \frac{-16 \pm \sqrt{576}}{-8}$$

$$x = \frac{-16 \pm 24}{-8}$$

$$x = \frac{-16 + 24}{-8}, x = \frac{-16 - 24}{-8}$$

The solutions are $x = -1, x = 5$.

Imaginary Numbers

There are many different sets of numbers, such as integers, natural numbers, rational numbers, and real numbers. There is also the set of imaginary numbers. This occurs when there is a negative number under a square root. The most basic imaginary number is $\sqrt{-1}$ and is notated as i. In other words, $i = \sqrt{-1}$. Operations using imaginary numbers are much like operations in the real number system.

Addition: $2i + 3i = 5i$

Subtraction: $7i - 3i = 4i$

When multiplying imaginary numbers, the exponents of the imaginary number are added together and their coefficients are multiplied.

Multiplication: $2i \cdot 5i = 10i^2$

Special properties are associated with the powers of i, as noted in the sidebar to the right.

Powers of i

$i = \sqrt{-1}$

$i^2 = (\sqrt{-1})^2 = -1$

$i^3 = i^2 \cdot i = -1 \cdot i = -i$

$i^4 = i^2 \cdot i^2 = -1 \cdot -1 = 1$

The powers continue to cycle following this pattern.

K Education & Technology

When dividing imaginary numbers, the exponents of the imaginary numbers are subtracted and their coefficients are divided.

Division: $\frac{49i^5}{7i^3} = 7i^2 = 7(-1) = -7$

Imaginary numbers can appear when working with square roots. A problem may ask for a solution to be written in simplest $a + bi$ form, which is a complex number. The a represents the real portion of the number, and the bi represents the imaginary portion of the number.

To simplify a complex number, first rewrite all radicals with a negative radicand as imaginary numbers by removing the negative sign and replacing it with the imaginary number i. Finally, simplify the radical.

Example

Solve the quadratic equation, and write your solution in simplest $a + bi$ form.

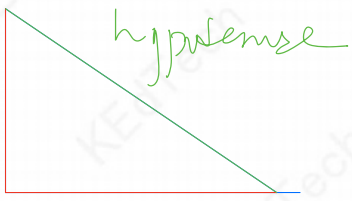
$$x^2 + 6x + 13 = 0$$

K Education & Technology

Right Triangle Trigonometry

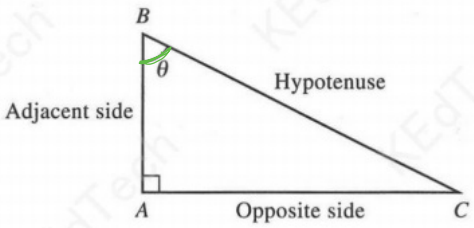
Right triangles are made up of a right angle and two acute angles. The sum of the 3 angles is 180°. The sides opposite the acute angles are called legs, and the side opposite the right angle is called the hypotenuse. The lengths of the sides of a right triangle must satisfy the Pythagorean theorem, $a^2 + b^2 = c^2$, where a and b are the legs and c is the hypotenuse.

The relationship between the angle measures and the side lengths is known as trigonometry. There are three trigonometric ratios of sides that correspond to an acute angle measure in a right triangle.



Three Trigonometric Ratios

Given the right triangle ABC, where angle A is the right angle shown in the figure, the sides are labeled based upon their relationship to angle θ.



Trigonometric Ratios of Angle θ

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{AC}{BC}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{AB}{BC}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{AC}{AB}$$

The ratio of sides in the box above can be remembered using the following mnemonic device:

$$S\frac{O}{H} \quad C\frac{A}{H} \quad T\frac{O}{A}$$

Special Right Triangles

Two triangles have known angle measures and corresponding side lengths. These are known as the special right triangles.

30°-60°-90° Triangle

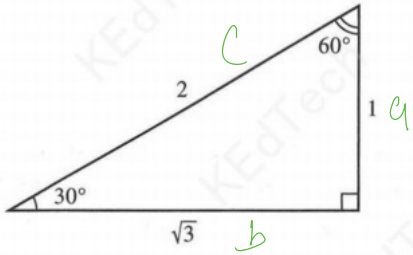
In this triangle, the two acute angles measure 30° and 60° and the third angle is the right angle, 90°. Their corresponding side lengths are in the ratio $1:\sqrt{3}:2$, as shown in the figure below.

As a result, below are six known trigonometric values that correspond to this special right triangle:

$$\sin 30° = \frac{1}{2} \qquad \cos 30° = \frac{\sqrt{3}}{2}$$

$$\tan 30° = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \qquad \sin 60° = \frac{\sqrt{3}}{2}$$

$$\cos 60° = \frac{1}{2} \qquad \tan 60° = \frac{\sqrt{3}}{1} = \sqrt{3}$$



$a^2 + b^2 = c^2$

$a = 1 \qquad c = 2$

$b = \sqrt{c^2 - a^2}$

$b = \sqrt{2^2 - 1^2}$

$= \sqrt{4 - 1}$

$= \sqrt{3}$

K Education & Technology

45°–45°–90° Triangle

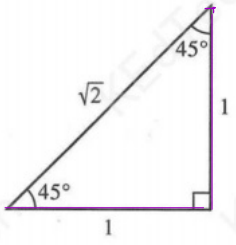
In this triangle, the two acute angles measure 45° and 45° and the third angle is the right angle, 90°. Their corresponding side lengths are in the ratio $1:1:\sqrt{2}$, as shown in the figure.

As a result, below are three known trigonometric values that correspond to this special right triangle:

$$\sin 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\cos 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\tan 45° = \frac{1}{1} = 1$$



The trigonometric values of angles other than 30°, 45°, and 60° can be measured using calculators. In this course, you will learn that the trigonometric values of angles that measure more than 90° can also be evaluated.

Law of Sines and Law of Cosines

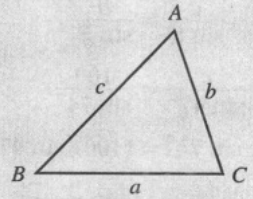
Two laws help determine the lengths and angles of triangles that don't necessarily have to be right triangles. They are known as the law of sines and the law of cosines.

Law of Sines

The law of sines relates the lengths of two sides of a triangle to the sines of the angles opposite those sides.

The Law of Sines

$$\frac{\text{side of } \triangle ABC}{\text{sine of angle opposite side}} = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



Example

In $\triangle ABC$, $a = 12$, $\sin A = 0.6$, and $\sin B = 0.4$. What is the length of side b?

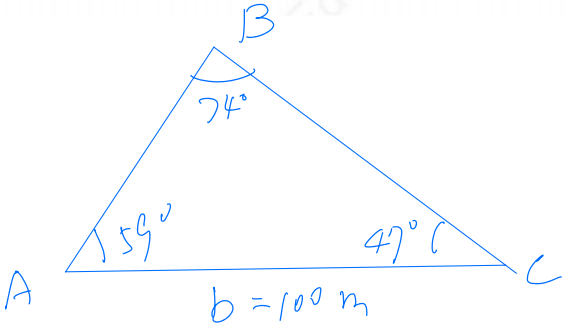
$$\frac{a}{\sin A} = \frac{b}{\sin B} \qquad \frac{12}{0.6} = \frac{b}{0.4}$$

$$b = \frac{12 \times 0.4}{0.6} = 8$$

K Education & Technology

Example

In ΔABC, $\angle A = 59°$, $\angle B = 74°$, and $b = 100$ meters. Find c to the nearest tenth of a meter.



$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$\angle C = 180° - \angle A - \angle B = 180° - 59° - 74°$

$\angle C = 47°$

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

$$c = \frac{b \cdot \sin C}{\sin B} = \frac{100 \times \sin 47°}{\sin 74°}$$

$c = 76.1$ m

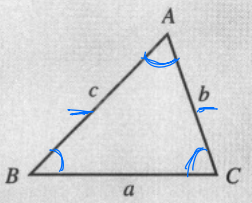
K Education & Technology

Law of Cosines

The law of cosines relates the lengths of the three sides of a triangle to the cosine of the angle opposite one of the sides.

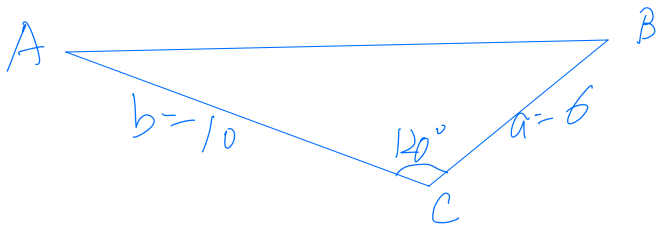
The Law of Cosines

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$



Example

In ΔABC, $a = 6$, $b = 10$, and $\angle C = 120°$. What is the length of side c?



$$c^2 = a^2 + b^2 - 2ab\cos 120°$$
$$= 6^2 + 10^2 - 2\times 6\times 10\times \cos 120°$$
$$= 196$$
$$c = \pm\sqrt{196} \qquad c = 14 \qquad c = -14\times$$

K Education & Technology

UNIT 1
Polynomial and Rational Functions

K Education & Technology

1

Rates of Change

Learning Objectives

In this chapter, you will learn:

- The difference between a relation and a function
- How to calculate the rate of change and its meaning
- How to calculate the rate of change of different functions

1.1 Relations and Functions

Relations

Most mathematical studies are based upon two different quantities and how they relate to one another. This is known as a relation. A relation can be represented by a diagram, an equation, or a list that specifies a specific relationship between groups of elements.

For example, consider relation r shown in Figure 1.1 as both a diagram and a list of ordered pairs.

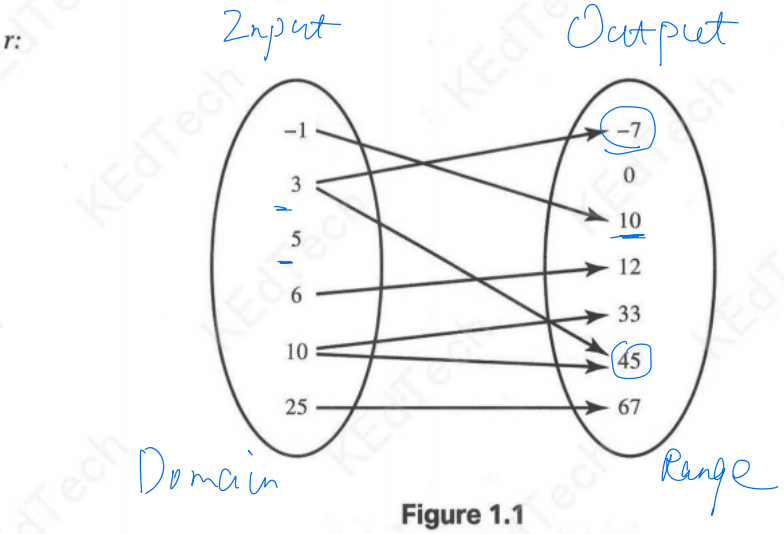


Figure 1.1

The elements in the oval on the left are known as the set of input values, which is also called the domain. The elements in the oval on the right are known as the set of output values, which is also called the range. The arrows lead from the inputs and end at their corresponding outputs.

Following the arrows helps to write the relation as a set of ordered pairs as shown below:

$$r: \{(-1, 10), (3, -7), (3, 45), (6, 12), (10, 33), (10, 45), (25, 67)\}$$

There are two things to notice about relation r:

1. Not all of the elements in the domain were used: 5 did not have a corresponding output.
2. Some inputs had multiple outputs: 3 mapped to −7 and 45, and 10 mapped to 33 and 45.

K Education & Technology

Functions

A relation that maps a set of input values to a set of output values where each input value is mapped to one and only one output value is defined as a function. For relation r as a function, 5 would need to be removed from the domain, and both 3 and 10 should be mapped to only one output each. This is shown in Figure 1.2.

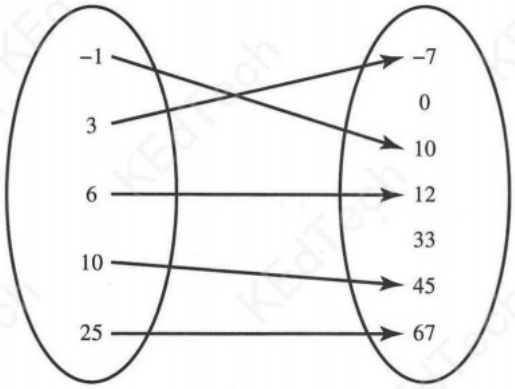


Figure 1.2

An example of a real-life model of a function is the cost of a taxi. Taxis calculate the cost of a ride based on the amount of time a passenger spends in the vehicle. Therefore, the cost of a particular ride is a function of the length of the ride. A passenger cannot have two charges for the same duration. In this example, time is the domain and cost is the range.

Functions are powerful mathematical tools for describing how one variable quantity depends on another. A function can be represented verbally, numerically, algebraically, or graphically. Variables are often used to represent the infinitely many possibilities for the domain and range of a function. The variable representing input values is called the independent variable, and the variable representing output values is called the dependent variable. Since functions, like relations, can be represented by ordered pairs, x is often used to represent the independent variable and y often represents the dependent variable. Then every input and its corresponding output can be represented by the ordered pair (x, y).

Often functions can be written as equations based upon a rule that describes the relationship between the independent and dependent variables. Returning to the cost of the taxi, for a passenger to ride, it costs \$2.00 to enter the car and \$1.50 for every minute of the ride. A table of values can be created to list the different corresponding inputs and outputs. For example, at 0 minutes, the cost is \$2.00 just to enter the taxi. After a minute of driving, \$1.50 is added to that cost for a total of \$3.50. At 2 minutes, another \$1.50 is added for a total cost of \$5.00. This is illustrated in Table 1.1.

Table 1.1

x = time (minutes)	y = cost (dollars)
0	2.00
1	3.50
2	5.00
3	6.50
4	8.00

Since there are infinitely many values that time can take on, listing all the corresponding pairs is tedious. It is more efficient to write the cost of a ride as an equation dependent upon time. Since the independent variable is time, let x represent the time in minutes and y represent the cost in dollars. The cost equation could be written as $y = 2.00 + 1.50x$.

K Education & Technology

There is another way to notate functions as equations. If f represents a function, then $f(x)$, read as "f of x," represents the value of the function when x takes on a specific value. In other words, $y = f(x)$. The benefit of this notation is that it allows you to see the two variables and clearly identify the independent and dependent variables. It is also a nice way to identify specific values. For example, to find the cost of the taxi ride after 5 minutes, we would input 5 for x to find the cost. This substitution would look like $f(5) = 2.00 + 1.50(5) = 9.50 = y$. After 5 minutes, the cost of the taxi would be \$9.50.

Different letters can be used to fit what the variables represent. For example, t can stand for time and c for cost. Our taxi equation can be rewritten as $c(t) = 2.00 + 1.50t$.

Example

If $f(x) = 10x - 2$, find the value of $f(3)$.

$f(3) = 10 \cdot 3 - 2 = 30 - 2 = 28$

Example

Given $g = \{(-1, 1), (0, 3), (4, 7), (5, 1), (8, -10)\}$, find $g(0)$.

Example

If $v(t) = t^2 + 5$, what is $v(c + 2)$?

$t = c + 2$

$(c+2)^2 + 5 = (c+2)(c+2) = c^2 + 2c + 2c + 4 + 5$

$= c^2 + 4c + 9$

Another way functions can be represented is by their graphs. To graph a function, plot its coordinate pairs as points on an xy-plane. The xy-plane is a coordinate grid with a horizontal line that represents the x-axis and a vertical line that represents the y-axis. Coordinate pairs are plotted by moving left or right on the x-axis and then moving down or up on the y-axis. If the function is continuous over its domain, the points of the pairs can be connected to reveal the graph. The graph provides a nice visual and can show different behaviors of the graph that can lead to important conclusions about the function.

Returning to the taxi cost function, $f(x) = 2.00 + 1.50x$, graph the function on the coordinate plane. Begin by plotting the values from Table 1.2 on the coordinate plane, as shown in Figure 1.3.

K Education & Technology

Table 1.2

x = time (minutes)	y = cost (dollars)
0	2.00
1	3.50
2	5.00
3	6.50
4	8.00

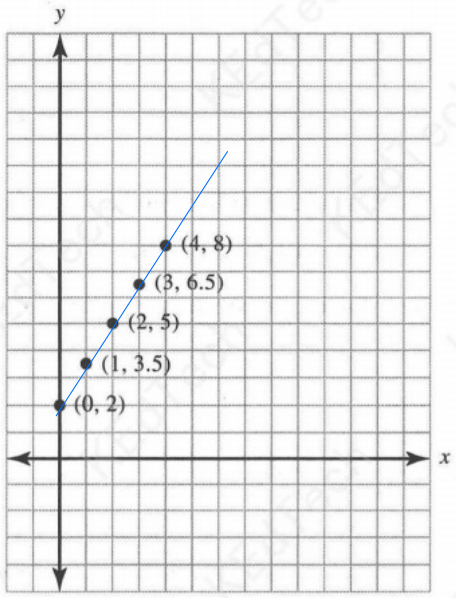


Figure 1.3

Since the domain is time, assume the inputs are all real numbers greater than zero. The points can be connected to create a continuous graph, as shown in Figure 1.4.

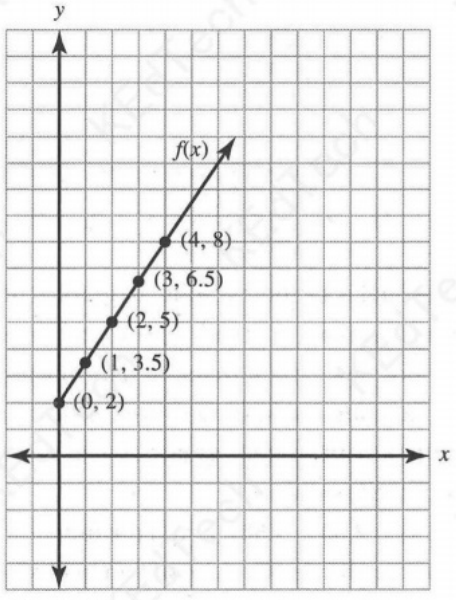


Figure 1.4

K Education & Technology

This is a linear function because its graph is a line. Note that the axes can change names, too, depending on the independent and dependent variables being used. If we graphed $c(t) = 2.00 + 1.50t$, the x-axis could be renamed as t and the y-axis could be renamed as either c or $c(t)$.

Various types of functions will be explored in later chapters of this book. Many have general concepts that can be applied, including domain, range, x- and y-intercepts, and graph behaviors.

Domain and Range

The domain of a function, $f(x)$, is the largest possible set of numbers x for which $f(x)$ is a real number. To find the domain graphically, look along the x-axis from left to right. The x-value of the point farthest to the left represents the lower bound of the domain, and the x-value of the point farthest to the right represents the upper bound of the domain.

The range of a function, $f(x)$, is the set of all values that y can have as x takes on each of its possible values.

Figure 1.5 is the graph of function $g(x)$.

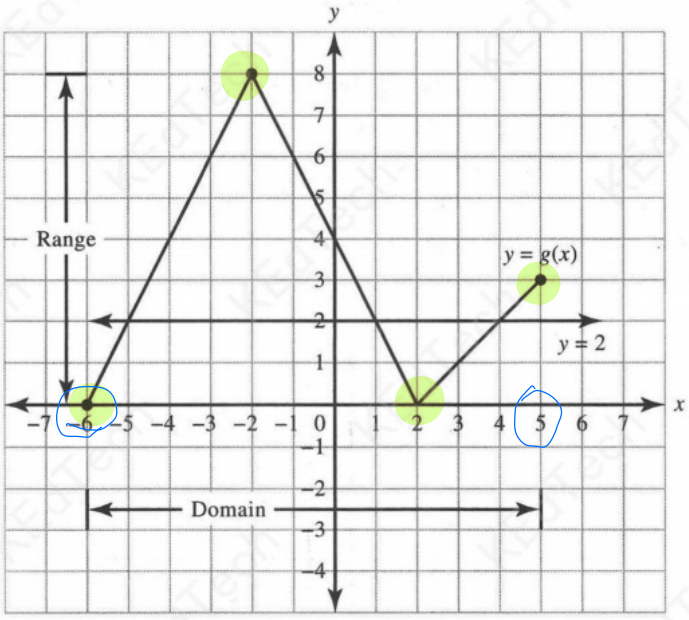


Figure 1.5

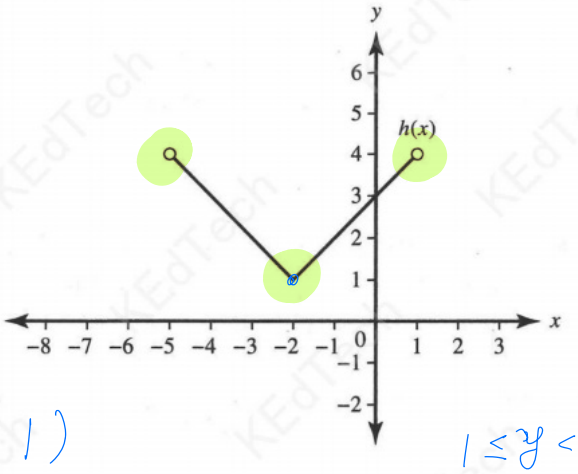
The point farthest to the left is $(-6, 0)$, and the point farthest to the right is $(5, 3)$. Therefore, the domain of $g(x)$ is $-6 \le x \le 5$, which can be written as the interval $[-6, 5]$.

The lowest points are $(-6, 0)$ and $(2, 0)$. The highest point is $(-2, 8)$. Therefore, the range of $g(x)$ is $0 \le y \le 8$, which can be written as the interval $[0, 8]$.

Figure 1.5 has closed circles on the endpoints, but not all graphs have closed endpoints, as seen in the next figure.

K Education & Technology

Figure 1.6 is the graph of function $h(x)$.



(−5, 1) $1 \le y < 4$

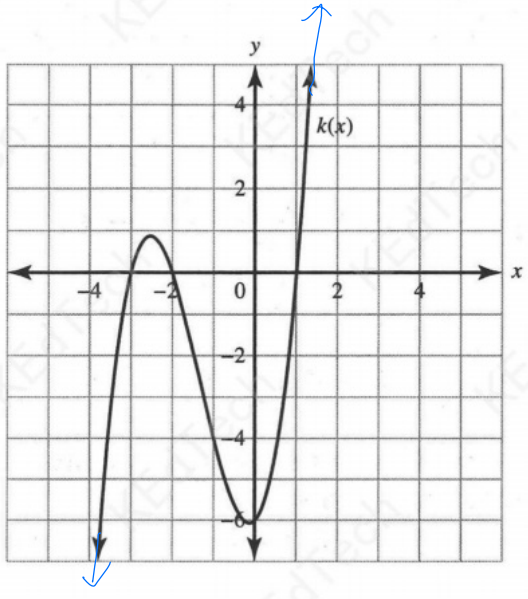
$-5 < x < 1$ [1, 4)

Figure 1.6

The point farthest to the left is $(-5, 4)$, and the point farthest to the right is $(1, 4)$. Since these points have open circles, it means the graph of the function goes up to those points but does not include those values. Therefore, the domain of $h(x)$ is $-5 < x < 1$, which can be written as the interval $(-5, 1)$.

The lowest point is $(-2, 1)$ and is graphed with a closed circle. The highest points are $(-5, 4)$ and $(1, 4)$, which are graphed with open circles. Therefore, the range of $h(x)$ is $1 \le y < 4$, which can be written as the interval $[1, 4)$.

Figure 1.7 is the graph of $k(x)$.



$x \in R$ $(-\infty, \infty)$

$y \in R$ $(-\infty, \infty)$

Figure 1.7

There are arrowheads on both endpoints. This means the graph continues in those directions forever. Since there is no left or right endpoint, the domain is all real numbers, which can be written in interval notation as $(-\infty, \infty)$. Since there is no lowest or highest endpoint, the range is all real numbers, which is written in interval notation as $(-\infty, \infty)$.

K Education & Technology

Example

State the domain and range of each of the functions in Figures 1.8 and 1.9.

1.

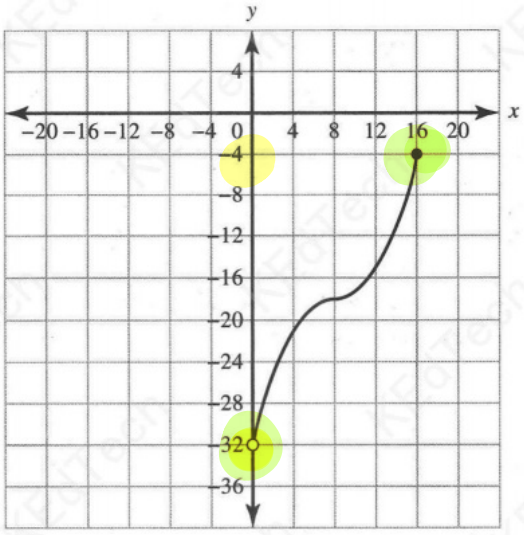


Figure 1.8

$0 < x \leq 16$

$(0, 16]$

$-32 < y \leq -4$

$(-32, -4]$

2.

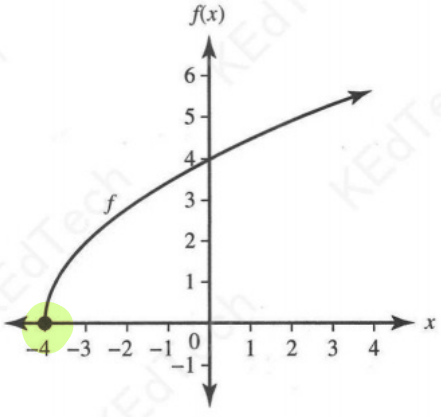


Figure 1.9

$x \geq -4$

$[-4, \infty)$

$y \geq 0$

$[0, \infty)$

K Education & Technology

Algebraically, the domain can be calculated by considering the input values that would make the function not exist or be undefined.

Rules to Find the Domain

There are two key rules to follow when algebraically finding the domain of a function.

1. Do not divide by zero. Exclude from the domain of a function any value of x that results in a 0 in the denominator. For example, if $f(x) = \frac{x+4}{x-3}$, then $x \neq 3$ since $f(3) = \frac{3+4}{3-3} = \frac{7}{0}$, which is undefined. Therefore, the domain of the function is the set of all real numbers except 3, which can be represented as $(-\infty, 3) \cup (3, \infty)$.
2. Do not take the square root (or any even root) of a negative number. Since the square root of a negative number is not a real number, the quantity underneath a square root must always be greater than or equal to 0. If $f(x) = \sqrt{x+5}$, then $x + 5 \geq 0$ and $x \geq -5$. The domain of the function is the set of all real numbers greater than or equal to -5, which can be represented as $[-5, \infty)$.

Intercepts

The x-intercept of the graph of a function is found where the graph intersects the x-axis. At this point, the output value is zero and the corresponding input values are said to be the zeros of the function. When graphically finding the zeros of a function, look to see where the graph intersects the x-axis.

The y-intercept of the graph of a function is found where the graph intersects the y-axis. At this point, the input value is zero.

› Example

Find the zeros of the function $f(x)$ shown in Figure 1.10. Also state the coordinates of the y-intercept.

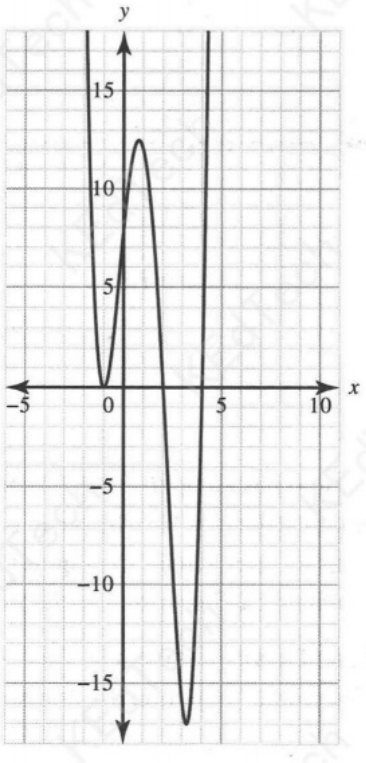


Figure 1.10

K Education & Technology

Example

Find the zeros of each of the following functions.

1. $f(x) = 2x - 48$
2. $g(x) = (x - 3)(x + 5)(x + 13)$

Example

Find the y-intercept of each of the following functions.

1. $f(x) = 2x - 48$
2. $g(x) = (x - 3)(x + 5)(x + 13)$

K Education & Technology

Notice in the previous two examples that a function can have multiple zeros or x-intercepts but only one y-intercept. This is true for every function since a function can have one and only one output for any input. Since the input for a y-intercept is always 0, that 0 can correspond to only one y-value in order to remain a function.

Remember that a function can have many x-intercepts or zeros but can have only one y-intercept.

This rule can easily be visualized using a graph and is called the vertical line test. The vertical line test states that an equation represents a function if no vertical line intersects the graph of the equation in more than one point.

The graph in Figure 1.11 represents a function since any vertical line that can be drawn intersects the graph in at most one point. The graph in Figure 1.12 is not a function since it fails the vertical line test.

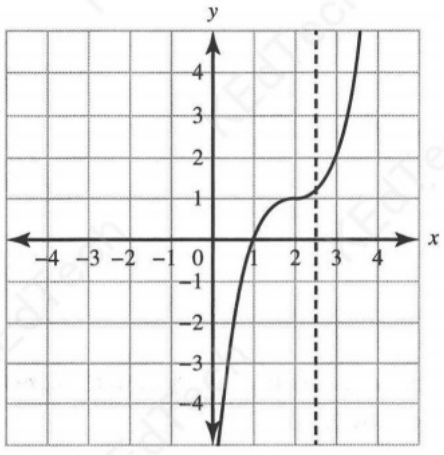


Figure 1.11

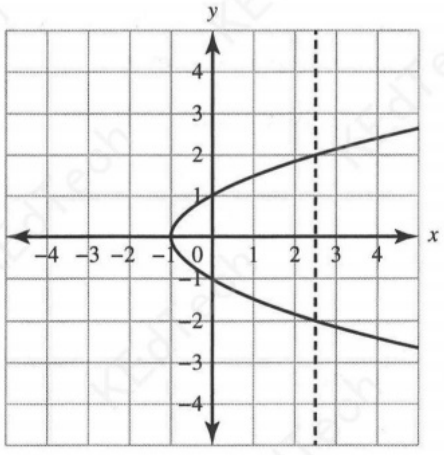


Figure 1.12

Additional Behaviors of Functions

The last general behaviors that will be discussed about functions can be easily viewed from a function's graph and help describe its shape. These behaviors can also be tested algebraically and are a major topic of conversation in calculus.

K Education & Technology

Increasing/Decreasing

A function is increasing over an interval of its domain if, as the input values increase, the output values always increase. In other words, for all a and b in the interval of the domain, if $a < b$, then $f(a) < f(b)$.

A perfect example of an increasing function is the taxi cost equation, $f(x) = 2.00 + 1.50x$. As time increases, the cost of the taxi continues to increase. As shown in Figure 1.4, as you follow the path of the graph from left to right, the graph continues to travel up.

Alternatively, a function is decreasing over an interval of its domain if, as the input values increase, the output values always decrease. In other words, for all a and b in the interval, if $a < b$, then $f(a) > f(b)$.

A real-world example that models a decreasing function is the value of a new car. After you buy a car, as time increases, the value of the car continues to decrease as the car is said to depreciate.

Figure 1.13 shows the graph of an increasing function, and Figure 1.14 shows the graph of a decreasing function.

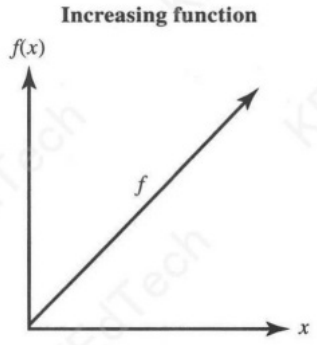


Figure 1.13

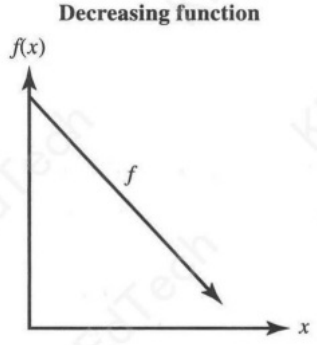


Figure 1.14

Note that a graph of a function can have both increasing and decreasing intervals over the domain.

Example

Using the graph of the function shown in Figure 1.15, identify the intervals where the function is increasing and the intervals where it is decreasing.

K Education & Technology

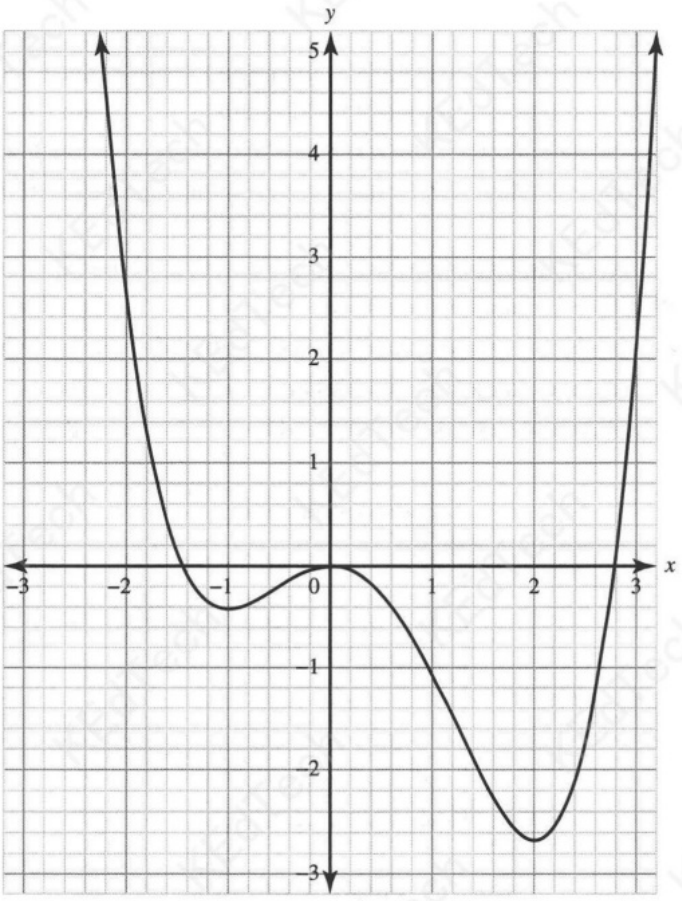


Figure 1.15

Concavity

Unlike linear functions, the graphs of other functions that will be discussed are not only continuous but are also smooth and appear to bend, sometimes upward or downward. This is known as the concavity of a graph, as shown in Figure 1.16.

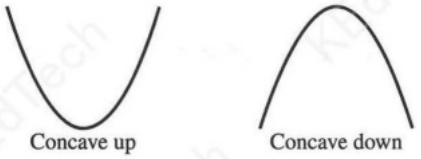


Figure 1.16

A more formal definition of concavity now leads to a discussion of rates of change.

K Education & Technology

1.2 Rates of Change

A rate is a comparison between two different quantities.

To find the average rate of change, either two points or two sets of inputs with their corresponding outputs are needed. Then the differences can be calculated.

Finding the average rate of change between an initial point and a final point is equivalent to finding the slope of the segment (or secant) between those points.

Average Rate of Change

The average rate of change is the change in the value of one quantity divided by the change in the value of another quantity. Common examples of average rate of change include finding the average speed $= \dfrac{\text{change in distance}}{\text{change in time}}$ and the slope of a line segment $= \dfrac{\Delta y}{\Delta x} = \dfrac{y_2 - y_1}{x_2 - x_1}$.

› Example

A car travels between two cities according to the distance-time graph shown in Figure 1.17.

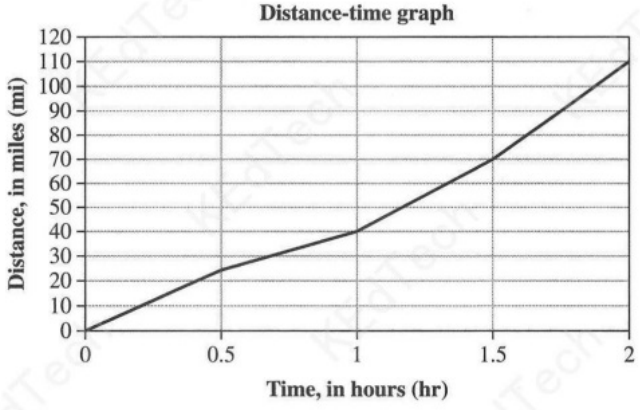


Figure 1.17

(a) What is the car's average speed for the first hour?
(b) What is the car's average speed for the entire trip?
(c) How does the average speed in part (a) compare with the average speed in part (b)?

K Education & Technology

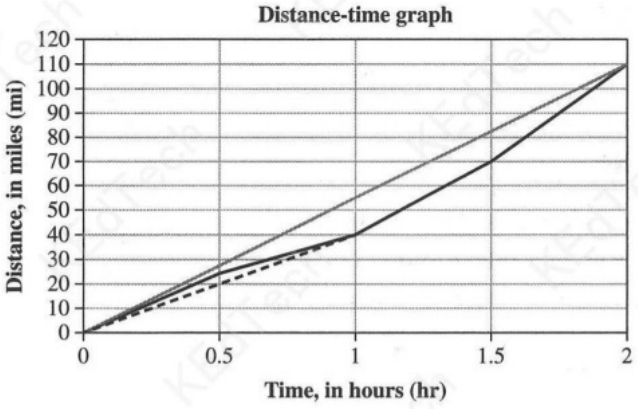


Figure 1.18

The average speed for the entire trip does not give all the details for the speeds along the way. The average speed is just that, an average or an overall idea of the rate of change between a starting point and an ending point.

Finding the speed at a specific instant in time is something entirely different. Graphically at one point, such as $t = 1$ hour, the secant line would change to a tangent line, as shown in Figure 1.19.

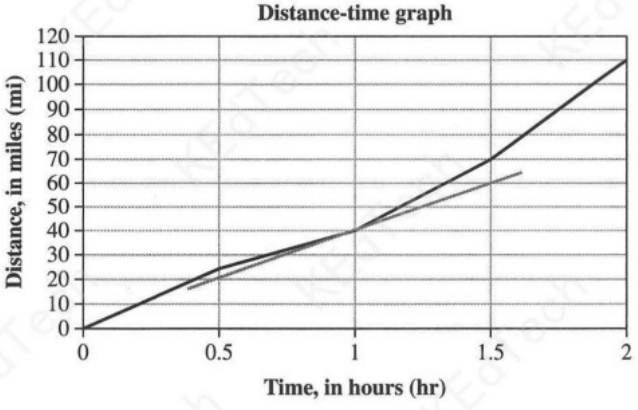


Figure 1.19

Finding the rate of change at this value is an issue since the previous slope formula needs two points and only one point is known for the tangent line.

The rate of change at a point can be approximated by the average rate of change of the function over small intervals containing the point, if such values exist.

K Education & Technology

Example

Given the graph of function $f(x) = x^2$ shown in Figure 1.20, find the average rate of change at point $P(4, 16)$.

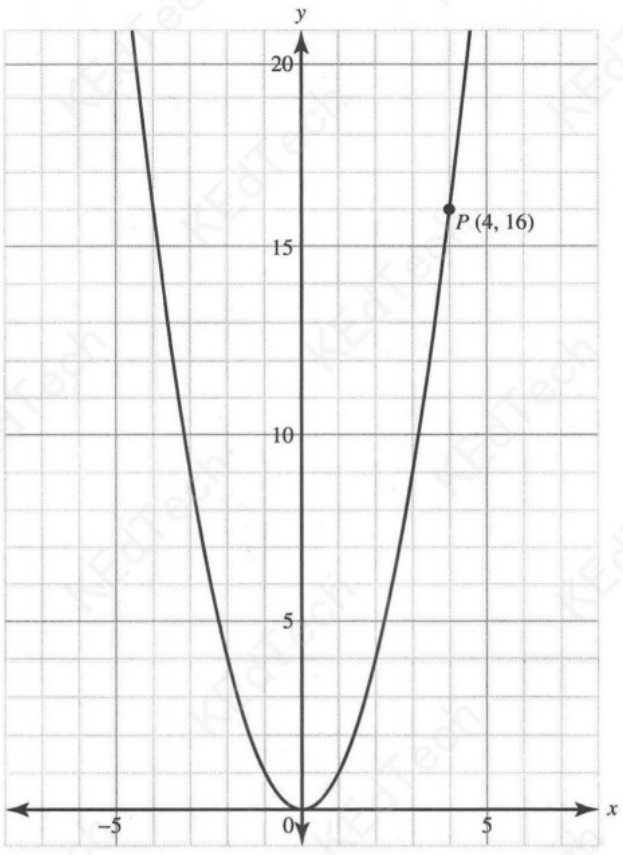


Figure 1.20

K Education & Technology

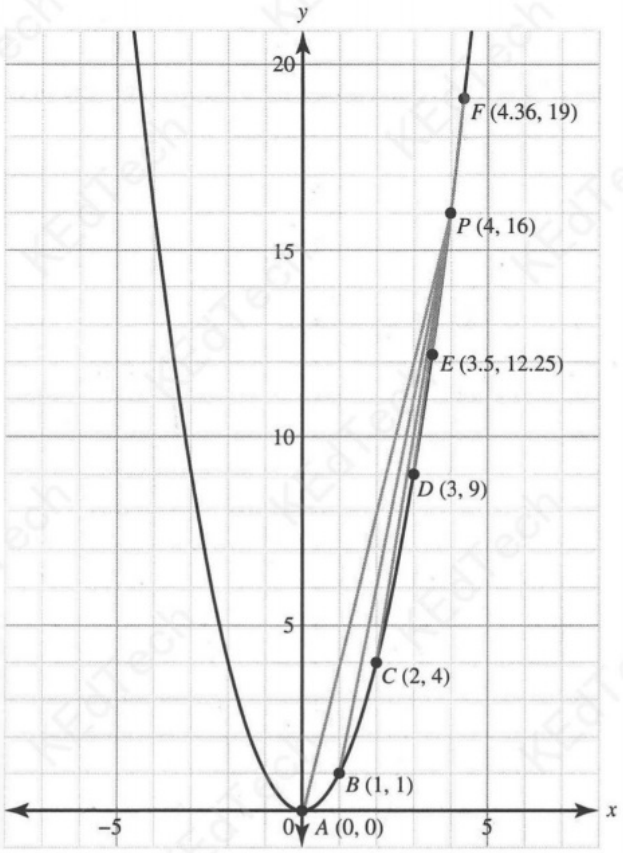


Figure 1.21

Table 1.3 will help to organize the different average rates of change.

Table 1.3

Point	P(4, 16)	Average Rate of Change
$A(0, 0)$	$P(4, 16)$	$\frac{16-0}{4-0}=\frac{16}{4}=4$
$B(1, 1)$	$P(4, 16)$	$\frac{16-1}{4-1}=\frac{15}{3}=5$
$C(2, 4)$	$P(4, 16)$	$\frac{16-4}{4-2}=\frac{12}{2}=6$
$D(3, 9)$	$P(4, 16)$	$\frac{16-9}{4-3}=\frac{7}{1}=7$
$E(3.5, 12.25)$	$P(4, 16)$	$\frac{16-12.25}{4-3.5}=\frac{3.75}{0.5}=7.5$
$F(4.36, 19)$	$P(4, 16)$	$\frac{16-19}{4-4.36}=\frac{-3}{-0.36}=8.\overline{333}$

For x-values smaller than 4, the slopes approach 7.5. For x-values greater than 4, the values approach $8.\overline{333}$.

Since the equation of the function is given, x-values that are even closer to 4 from both the left and right directions can be substituted into the function as shown in Table 1.4.

K Education & Technology

Table 1.4

Point	$P(4, 16)$	Average Rate of Change
$G(3.9, 15.21)$	$P(4, 16)$	$\frac{16 - 15.21}{4 - 3.9} = \frac{0.79}{0.1} = 7.9$
$H(3.95, 15.6025)$	$P(4, 16)$	$\frac{16 - 15.6025}{4 - 3.95} = \frac{0.3975}{0.05} = 7.95$
$I(3.99, 15.9201)$	$P(4, 16)$	$\frac{16 - 15.9201}{4 - 3.99} = \frac{0.0799}{0.01} = 7.99$
$J(4.001, 16.008001)$	$P(4, 16)$	$\frac{16 - 16.008001}{4 - 4.001} = \frac{-0.008001}{-0.001} = 8.001$
$K(4.01, 16.0801)$	$P(4, 16)$	$\frac{16 - 16.0801}{4 - 4.01} = \frac{-0.0801}{-0.01} = 8.01$
$L(4.1, 16.81)$	$P(4, 16)$	$\frac{16 - 16.81}{4 - 4.1} = \frac{-0.81}{-0.1} = 8.1$

Coming in from the left and getting closer to point $P(4, 16)$, points G, H, and I have slopes that approach 7.99. Coming in from the right and getting closer to point $P(4, 16)$, points J, K, and L have slopes that approach 8.001. Overall, it seems that the slopes are approaching the value of 8. Therefore, the approximate average rate of change at point P is 8.

The rates of change at two points can be approximated by calculating average rate of change approximations over sufficiently small intervals containing each point, if such values exist.

Example

A small local restaurant sponsored a contest to name their new special dish. Patrons submitted their entries between noon ($t = 0$) and 10 P.M. ($t = 10$). The number of entries t hours after noon is modeled by the function N for $0 \leq t \leq 10$. Values of $N(t)$, at various times t, are shown in Table 1.5. Use the data in the table to approximate the rate at which entries were being deposited at time $t = 8$.

Table 1.5

t (hours)	0	3	6	7	9	10
$N(t)$ (number of entries)	0	25	43	50	79	125

A positive rate of change indicates that as one quantity increases or decreases, the other quantity does the same. Algebraically, when calculating the average rate of change, the value simplifies to a positive number. In the previous example, as time increased from $t = 0$ to $t = 10$, the number of entries continuously increased from $N = 0$ to $N = 125$. In other words, the rate at which entries were being deposited increased as the day went by. A positive rate of change can be seen on the graph shown in Figure 1.22.

K Education & Technology

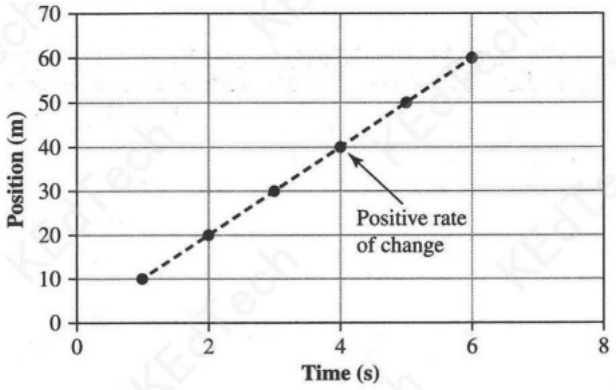


Figure 1.22

A negative rate of change indicates that as one quantity increases, the other decreases. For example, a freshman in college begins the semester with \$1,000 in their savings account. Each week they take out \$75 to pay for their expenses. After one week, the student has \$925. The average rate of change is $\frac{925-1000}{1-0}=\frac{-75}{1}=-75$. Graphically, a negative rate of change can be seen in Figure 1.23.

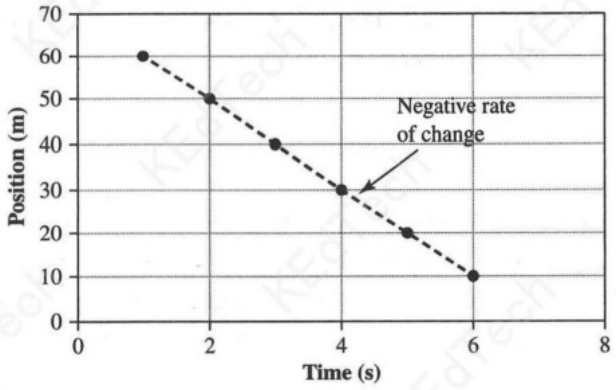


Figure 1.23

Unlike Figures 1.22 and 1.23, not all graphs are linear. Each type of function that will be discussed in later chapters has its own unique graphical behaviors. One such behavior is known as concavity. The concavity of a function describes how the graph of the function curves and bends as shown in Figure 1.24.

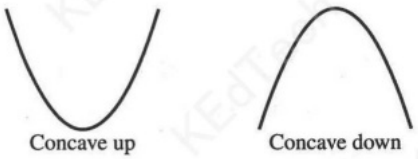


Figure 1.24

More formally, the graph of a function is concave up on intervals in which the rate of change is increasing. To illustrate this, consider the function $f(x)=(x-4)^2+2$, as shown in Figure 1.25 and Table 1.6.

K Education & Technology

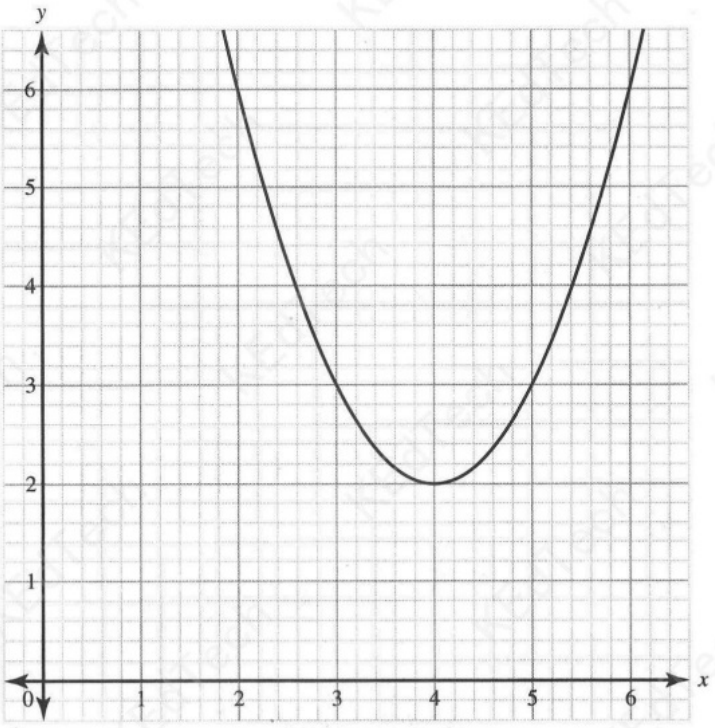


Figure 1.25

Table 1.6

x	$f(x)$
2	6
3	3
4	2
5	3
6	6

Table 1.7 calculates the average rates of change between the pairs of points.

Table 1.7

Point 1	Point 2	Average Rate of Change
(2, 6)	(3, 3)	−3
(3, 3)	(4, 2)	−1
(4, 2)	(5, 3)	1
(5, 3)	(6, 6)	3

As x is increasing, the average rate of change is also increasing. This illustrates why the graph of the function is concave up in Figure 1.25.

Alternatively, the graph of a function is concave down on intervals in which the rate of change is decreasing. To illustrate this, consider the function $f(x) = -(x - 4)^2 + 2$, as shown in Figure 1.26 and Table 1.8.

K Education & Technology

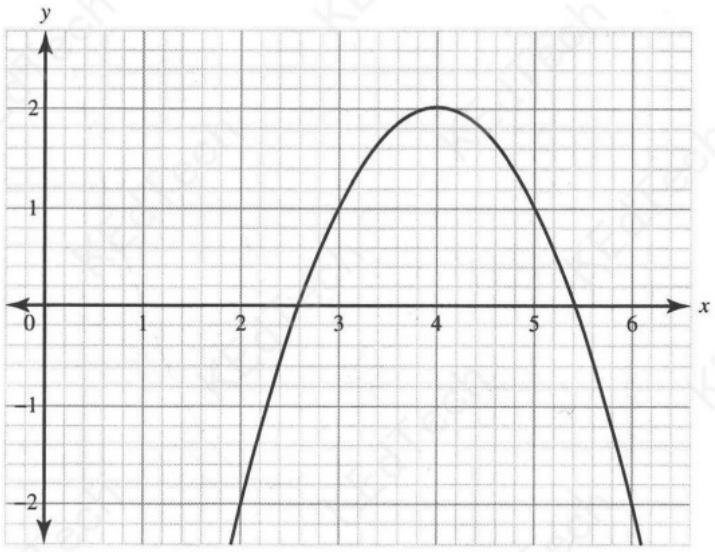


Figure 1.26

Table 1.8

x	$f(x)$
2	−2
3	1
4	2
5	1
6	−2

Table 1.9 calculates the average rates of change between the pairs of points.

Table 1.9

Point 1	Point 2	Average Rate of Change
(2, −2)	(3, 1)	3
(3, 1)	(4, 2)	1
(4, 2)	(5, 1)	−1
(5, 1)	(6, −2)	−3

As x is increasing, the average rate of change is decreasing. This illustrates why the graph of the function is concave down in Figure 1.26.

K Education & Technology

1.3 Rates of Change in Linear Functions

Many different types of functions will be discussed and analyzed in this course. Calculating the average rate of change can be used to help identify some of them. Consider Table 1.10, which shows the ordered pairs of the linear function $y = f(x) = -4x + 7$.

Table 1.10

x	$y = f(x)$	Average Rate of Change $= \frac{\Delta y}{\Delta x}$
-2	15	$= \frac{11 - 15}{-1 - (-2)} = \frac{-4}{1} = -4$
-1	11	$= \frac{7 - 11}{0 - (-1)} = \frac{-4}{1} = -4$
0	7	$= \frac{3 - 7}{1 - 0} = \frac{-4}{1} = -4$
1	3	$= \frac{-1 - 3}{2 - 1} = \frac{-4}{1} = -4$
2	-1	$= \frac{-5 - (-1)}{3 - 2} = \frac{-4}{1} = -4$
3	-5	

As the value of x increases by 1, the value of y decreases by 4. In other words, for every consecutive integer ordered pair, the average rate of change is -4. This would also be true over any interval length. Consider the ordered pairs $(-2, 15)$ and $(3, -5)$. The average rate of change is

$$\frac{\Delta y}{\Delta x} = \frac{-5 - 15}{3 - (-2)} = \frac{-20}{5} = -4$$

It is not a coincidence that the average rate of change of the linear function $f(x) = -4x + 7$ is the slope of the linear function.

Rate of Change

In general, all linear functions have a constant rate of change, $m = \frac{\Delta y}{\Delta x}$, and can be written in the slope-intercept form $y = f(x) = mx + b$, where b is the y-coordinate of the y-intercept. Only linear functions have a constant average rate of change. The average rate of change can be used to determine whether a particular function is linear or not. Functions that are not linear are said to be nonlinear.

Example

A strain of bacteria is placed into a Petri dish and allowed to grow. The data collected are shown in Table 1.11. The population is measured in kilograms and the time in hours. The data in Table 1.12 represent the maximum number of hours an individual should work during the week. For both tables, use the average rate of change to determine whether the function is linear. Graph each to check.

(a)

Table 1.11

Time (hours), x	Population (kg), y
0	0.09
1	0.12
2	0.16
3	0.22
4	0.29
5	0.39

K Education & Technology

(b)

Table 1.12

Age, x	Time (hours), y
20	50
30	47.5
40	45
50	42.5
60	40
70	37.5

K Education & Technology

K Education & Technology

1.4 Rates of Change in Quadratic Functions

A quadratic function is a function of the form $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers and $a \neq 0$. The domain of a quadratic function is the set of all real numbers.

As seen in the previous section, the rate of change of a linear function is constant. The rate of change of a quadratic function is not constant; there are no straight-line segments on a parabola, which is the graph of a quadratic function. To calculate the rate of change, determine the average rate of change by finding the slope of the segment that connects two points on the parabola. This segment is called the secant line.

K Education & Technology

Example

Consider the quadratic function $y = x^2$ and its graph shown in Figure 1.29.

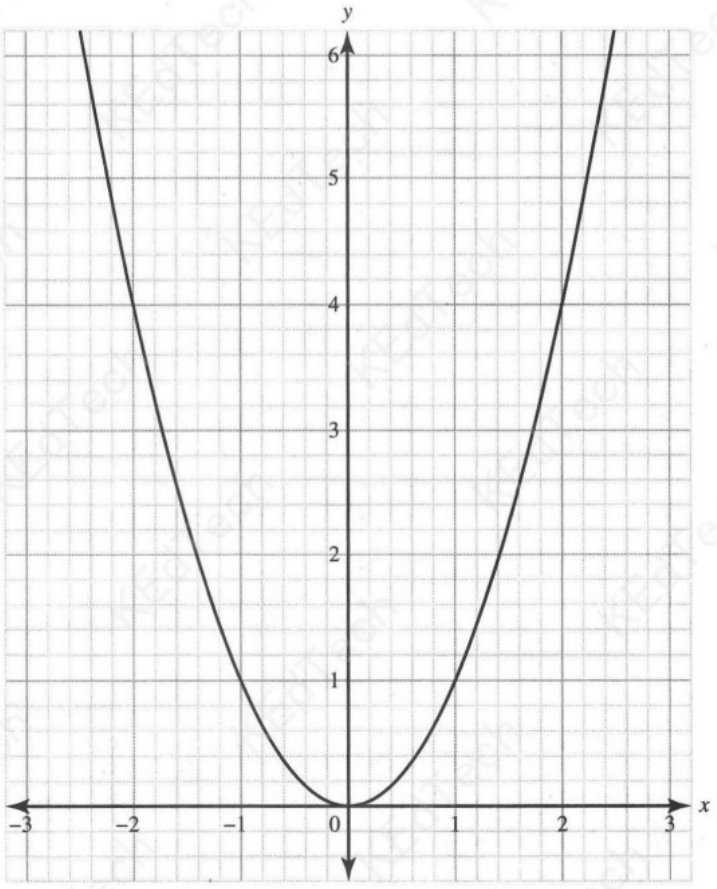


Figure 1.29

Table 1.15 shows the average rate of change between points that have consecutive, even-length input values.

Table 1.15

Points	Average Rate of Change = Slope of Secant Line
(0, 0) and (1, 1)	$= \frac{1-0}{1-0} = 1$
(1, 1) and (2, 4)	$= \frac{4-1}{2-1} = 3$
(2, 4) and (3, 9)	$= \frac{9-4}{3-2} = 5$

K Education & Technology

The graph in Figure 1.30 gives a visual representation of the slope of the secant lines.

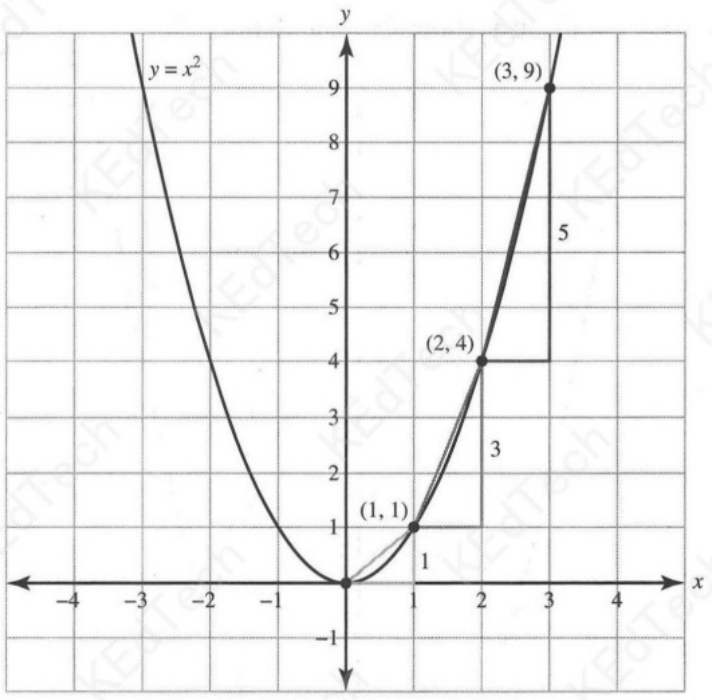


Figure 1.30

As x increases, we can see that the average rates of change are increasing, which coincides with the graph of the parabola being concave up. More specifically, since x increases by the equal interval of 1, we can see a more specific pattern emerging with the slopes of the secant lines. The rate of change of the average rates of change of this quadratic function is constant, specifically the constant 2. This will happen with any quadratic function.

Example

Which of the following functions represents a quadratic equation? State if the graph of the parabola would be concave up or concave down.

(a)

Table 1.16

x	$f(x)$
−3	−30
−2	−14
−1	−4
0	0
1	−2
2	−10

K Education & Technology

(b)

Table 1.17

x	$g(x)$
-3	76
-2	55
-1	36
0	19
1	4
2	-9

(c)

Table 1.18

x	$h(x)$
-3	81
-2	16
-1	1
0	0
1	1
2	16

K Education & Technology

K Education & Technology

Multiple-Choice Questions

1. Which of the following functions has a domain of $x > 0$?

   (A) $f(x) = x^2$
   (B) $f(x) = \sqrt{x}$
   (C) $f(x) = \log x$
   (D) $f(x) = e^x$

2. Which of the following are functions?

   I. $y - x = 17 + 2x$
   II. $x^2 + y^2 = 64$
   III. $x - y^2 = 81$
   IV. $x^2 + y = 4$

   (A) I only
   (B) I and III only
   (C) I and IV only
   (D) I, II, III, and IV

3. Find the average rate of change of $f(x) = 3x^2 - 2$ from $x = 2$ to $x = 6$.

   (A) 24
   (B) 18
   (C) $\frac{1}{18}$
   (D) $\frac{1}{24}$

4. Which of the following linear functions is increasing?

   (A) $s(t) = 4 - \frac{1}{2}t$
   (B) $h(z) = 10$
   (C) $g(x) = -3x + 2$
   (D) $f(x) = -7 + 0.9x$

5. If a represents the zero of the function $f(x) = (x - 3)^2$, evaluate $g(a)$ if $g(x) = 10 - 2(x - 1)^3$.

   (A) -6
   (B) -3
   (C) 1
   (D) 3

6. As a pot of boiling water cools, the temperature of the water is recorded and placed in the table below, where time t is measured in minutes and temperature $F(t)$ is measured in degrees Fahrenheit. Using the data in the table, which of the following best approximates the rate at which the temperature of the water is changing at time $t = 3.5$?

t (minutes)	0	2	5	9	10
$F(t)$ (degrees Fahrenheit)	150.8	140	125.6	111.2	109.4

   (A) -5.04°F/min
   (B) -4.8°F/min
   (C) -4.4°F/min
   (D) -4.14°F/min

7. Given the table of values below, which of the following best describes the behavior of the graph of the function in the interval $1 \leq x \leq 6$?

x	1	2	3	4	5	6
y	11	6	-5	-22	-45	-74

   (A) increasing, concave down
   (B) increasing, concave up
   (C) decreasing, concave down
   (D) decreasing, concave up

8. Cailey loves to bake and decides to open her own cookie stand, named Cailey's Cookies. Her daily fixed costs are \$120 a day, and each cookie costs \$0.99 to bake. Which of the following represents a model that expresses the cost, C, of baking m cookies in a day?

   (A) $C(m) = 0.99m + 120$
   (B) $C(m) = 0.99x + 120$
   (C) $C(m) = 120m + 0.99$
   (D) $C(m) = 0.04125m + 5$

K Education & Technology

9. For the function $f(x) = x^2 - 4x + 1$, evaluate $f(x + h)$.

   (A) $x^2 - 4x + 1 + h$
   (B) $h^2 - 4h + 1$
   (C) $x^2 + h^2 + 2xh - 4x - 4h + 1$
   (D) $x^2 + h^2 - 4x - 4h + 1$

10. Which of the following functions has the domain $(-\infty, \infty)$, $x \neq -9$, $x \neq 4$?

    (A) $f(x) = \sqrt{x - 36}$
    (B) $g(x) = \dfrac{x - 3}{x^2 + 5x - 36}$
    (C) $h(x) = \dfrac{x}{(x + 4)(x - 9)}$
    (D) $i(x) = x^2 + 5x - 36$

K Education & Technology