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Exponential vs. Linear Growth

The concepts in this chapter build on what we learned earlier in the percents chapter. Let's see how.

EXAMPLE 1: Jonas has a savings account that earns 3 percent interest compounded annually. His initial deposit was \$800. Which of the following expressions gives the value of the account, in dollars, after 10 years?

A) $800(1.30)^{10}$ B) $800 + 30(10)$ C) $800(1.03)(10)$ D) $800(1.03)^{10}$

A 3 percent interest rate compounded annually means Jonas earns 3 percent on the account once a year. This isn't just 3% on the original \$800. This is 3% on whatever's in his account at the time, including any interest he's already earned in previous years. So if we're in year 5, Jonas would earn 3% interest on the original \$800 and 3% interest on the total interest accumulated in years one through four. This interest that's earned on top of interest is what defines **compound interest**.

If we try to calculate the interest earned in the account after each and every year, this problem would take forever. Instead, let's take what we learned from the percents chapter and apply it here. As we know, a 3% increase is equivalent to a multiplication factor of 1.03. Therefore, the total value of the account after each year is just 1.03 times whatever the value was last year:

End of Year 1 total: $800(1.03) = 800(1.03)^1$
End of Year 2 total: $800(1.03)(1.03) = 800(1.03)^2$
End of Year 3 total: $800(1.03)(1.03)(1.03) = 800(1.03)^3$
End of Year 4 total: $800(1.03)(1.03)(1.03)(1.03) = 800(1.03)^4$

See the pattern in the exponents? The value of the account after t years is just $800(1.03)^t$. Therefore, the value of the account after 10 years is $800(1.03)^{10}$, answer $\boxed{(D)}$. No calculations needed.

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EXAMPLE 2: The population of an ant colony doubles every month. If there were 500 ants in the colony on May 1st, which function represents the population of the ant colony, $A(t)$, t months after May 1st?

A) $A(t) = 500(2t)$ B) $A(t) = 2(500)^t$ C) $A(t) = 500 + 2t$ D) $A(t) = 500(2)^t$

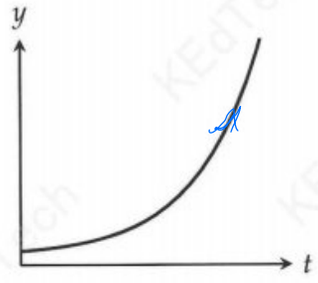
Since the population doubles every month, the population after the first month is 500(2). The population after the second month is double the population after the first month: $500(2)(2) = 500(2)^2$. The population after the third month is double the population after the second month: $500(2)(2)(2) = 500(2)^3$. See where this is going? The pattern is exactly the same as in Example 1 except the initial value here is 500 and the multiplication factor is 2. Therefore, the ant population after t months is $500(2)^t$, answer $\boxed{(D)}$.

A bank account earning 3 percent every year and an ant population doubling every month are examples of **exponential growth**, which occurs when a value grows periodically by a factor greater than 1. As we've seen, exponential growth can be modeled as an equation that looks like

$$y = ab^t$$

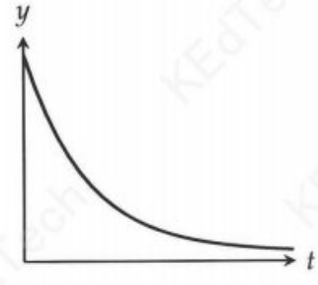
where y is the value after t specified units of time (e.g. years), a is the initial value, and b is a growth factor greater than one. The initial value a is sometimes referred to as the **coefficient** and the growth factor b is sometimes referred to as the **base**. In our bank account example, 800 was the initial value and 1.03 was the growth factor. In our ant population example, 500 was the initial value and 2 was the growth factor.

Graphs of exponential growth have the following shape:



Notice how the graph creeps up slowly at first but then shoots up faster and faster over time. That's exponential growth.

Exponential decay, however, is the opposite. Imagine a radioactive substance that loses mass over time. It loses a lot of its mass in the beginning and then loses it more and more slowly as time goes by.



It's worth memorizing the shapes of these graphs of exponential growth and decay. The SAT may test you explicitly on them.

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The equation for exponential decay is the same as the equation for exponential growth:

$$y = ab^t$$

The only difference is that the growth factor, b, is less than 1. So an equation that models exponential decay might look like

$$y = 400(0.6)^t \qquad y = ab^x \qquad b < 1$$

where y is the mass, in grams, of a radioactive substance t years from now. The 400 indicates that the substance currently has a mass of 400 grams, and the 0.6 indicates that at the end of each year, the substance is left with 60% of the mass it started the year with. In other words, the substance loses 40% of its mass each year.

So far, the examples we've discussed have been relatively simple. To model more complicated cases of exponential growth and decay, such as a bank account growing by 3% every 2 years or a radioactive substance losing half its mass every 9 months, we'll need to use a more generalized equation, one that you absolutely need to memorize:

$$y = ab^{\frac{t}{k}}$$

where y is the value after t specified units of time, a is the initial value, and k is the time required for the value of y to change by a factor of b. Note that t and k must be in the same units. So if t is in years, k should also be in years.

Let's go over some examples so that you fully understand what k means.

EXAMPLE 3:

$$M(t) = 400(1.05)^{\frac{t}{3}}$$

The function M models the mass, in nanograms, of a particle after t years. Which of the following is the best interpretation of $(1.05)^{\frac{t}{3}}$ in this context?

A) The mass of the particle increases by 5% every 4 months.
B) The mass of the particle increases by 5% every 3 years.
C) The mass of the particle increases by 5 nanograms every 4 months.
D) The mass of the particle increases by 5 nanograms every 3 years.

We have an exponential equation with $k = 3$ and $b = 1.05$, which means it takes 3 years for the mass of the particle to change by a factor of 1.05. In other words, the particle's mass increases by 5% every 3 years. And indeed, if we plug in $t = 3$, we get $400(1.05)^1 = 400(1.05)$, which represents one 5% increase after 3 years. Answer $\boxed{(B)}$.

Let's see why the generalized equation $y = ab^{\frac{t}{k}}$ works by taking a deeper look at this example. The mass of the particle changes by a factor of 1.05 for every increase of 1 in the value of the exponent, $\frac{t}{3}$. If we start the value of the exponent at 0, then $\frac{t}{3} = 0$. Multiplying both sides by 3 yields $t = 0$ years. Now if we set the value of the exponent to 1, then $\frac{t}{3} = 1$. Multiplying both sides by 3 yields $t = 3$ years. Since the value of the exponent increases by 1 from $t = 0$ years to $t = 3$ years, 3 years must be the time required for the particle's mass to change by a factor of 1.05. Repeat these steps with k instead of 3, and you'll have proven why k represents the time required for the value of y to change by a factor of b in the generalized equation.

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